New supersymmetric solutions of $\mathcal{N} = 2$, $D = 5$ gauged supergravity with hyperscalars

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ABSTRACT: We construct new supersymmetric solutions, including AdS bubbles, in an $\mathcal{N} = 2$ truncation of five-dimensional $\mathcal{N} = 8$ gauged supergravity. This particular truncation is given by $\mathcal{N} = 2$ gauged supergravity coupled to two vector multiples and three incomplete hypermultiplets, and was originally investigated in the context of obtaining regular AdS bubble geometries with multiple active R-charges. We focus on cohomogeneity-one solutions corresponding to objects with two equal angular momenta and up to three independent R-charges. Curiously, we find a new set of zero and negative mass solitons asymptotic to $AdS_5/\mathbb{Z}_k$, for $k \geq 3$, which are everywhere regular without closed timelike curves.

Contents

1. Introduction

While sphere compactifications of string theory and M-theory have been known for many years, they have taken on renewed importance since the advent of the AdS/CFT correspondence [1]. In particular, an extremely well studied system is that of type IIB string theory on $\mathrm{AdS}_5 \times S^5$ and the dual $\mathcal{N} = 4$ super-Yang Mills gauge theory. In the supergravity limit, this system can be investigated from both ten-dimensional and five-dimensional perspectives, with the latter corresponding to type IIB supergravity compactified on S^5. When consistently truncated, this yields $\mathcal{N} = 8$ gauged supergravity in five dimensions. Although this theory is generally well understood, in many cases it is possible to further simplify the system by invoking an $\mathcal{N} = 2$ subsector of the full theory which retains the $\mathcal{N} = 2$ supergravity multiplet coupled to two abelian vectors (often denoted the 'STU model').

Three-charge black hole solutions in the STU model were first obtained in [2, 3] as AdS generalizations of asymptotically Minkowskian R-charged black holes. However, in the BPS limit these solutions in fact develop naked singularities; they have been called superstars in [4] because of their relation to distributions of giant gravitons. Subsequently, genuine BPS black holes were obtained by Gutowski and Reall in [5, 6] by the addition of two equal angular momenta, which were generalised to have arbitrary angular momenta in [7, 8]. From a supergravity point of view, this provides an explicit de-singularization of the superstar by turning on rotation.

Following the work of Lin, Lunin and Maldacena on bubbling AdS [9], it was shown that the 1/2 BPS superstar, corresponding to specifying an intermediate boundary value on the AdS disk, may be desingularized by an alternative distortion of the AdS disk into an ellipse. The resulting 'AdS bubble' solution (including the three-arbitrary charge generalization) was presented in [10], and involves additional $\mathcal{N} = 8$ scalar excitations which lie outside the conventional $\mathcal{N} = 2$ truncation. Furthermore, these AdS bubble solutions are horizon-free and everywhere regular. (The one, two and three charged bubbles preserve 1/2, 1/4 and 1/8 of the supersymmetries, and may be described by ellipsoidal droplets in the generalized LLM phase space [11].)

Although these two methods for avoiding singularities are rather distinct (one uses rotation to generate a horizon, while the other has no horizon, but requires going beyond the STU model), they both apply to the same system of IIB supergravity on $\mathrm{AdS}_5 \times S^5$. Thus, in this paper we wish to develop a unified framework for describing all of the above BPS solutions in a five-dimensional supergravity context. In order to do so, we have to add three additional scalars, φ_I with $I = 1, 2, 3$, to the STU model. These scalars arise naturally from the diagonal elements of the $\mathrm{SL}(6, \mathbb{R})/\mathrm{SO}(6)$ coset of the S^5 reduction of IIB supergravity to $\mathcal{N} = 8$ in five dimensions. Viewed from a purely $\mathcal{N} = 2$ perspective, these scalars reside within three hypermultiplets.[1] The unified picture we use is then that

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[1]However, these are incomplete hypermultiplets, as we ignore their other components. This suffices for our present purposes, since we wish to study supersymmetric configurations in which the other components of the supermultiplets vanish.

of the STU model ($\mathcal{N} = 2$ gauged supergravity with two vector multiplets) coupled to three incomplete hypermultiplets.

The Gutowski-Reall black holes [5, 6] were obtained using the G-structure (invariant tensor) method of constructing supersymmetric solutions. This method was initially developed for minimal $\mathcal{N} = 2$ supergravity in four dimensions [12, 13], and subsequently applied to minimal ungauged [14] and gauged [15] $\mathcal{N} = 2$ supergravities in five dimensions. One advantage of the G-structure method is that it leads to a full classification (as well as an implicit construction) of *all* backgrounds admitting at least one Killing spinor. In this way, one could in principle obtain a complete understanding of all regular solutions of the STU model coupled to hypermatter scalars φ_I, with or without horizon. In practice, however, the invariant tensor construction which arises for this model is predicated on the choice of an appropriate four-dimensional Kähler base upon which the rest of the solution is built. This choice of base leads to an extremely rich structure of solutions, as can be witnessed from all the recent developments in constructing new BPS black holes and black rings in five dimensions.

In this paper, we limit ourselves to a cohomogeneity-one base with bi-axial symmetry, which preserves $\mathrm{SU}(2)_L \times \mathrm{U}(1) \subset \mathrm{SU}(2)_L \times \mathrm{SU}(2)_R \simeq \mathrm{SO}(4)$ isometry. This is sufficient to obtain all known black holes and AdS bubbles with two equal rotations turned on. Curiously, however, the isometry of the base is not required by the supersymmetry analysis to extend to that of the full solution, a fact which was also noted in [16] in the context of cohomogeneity-two solutions. As part of our analysis, we find that solutions with the full $\mathbb{R} \times \mathrm{SU}(2)_L \times \mathrm{U}(1)$ isometry in five dimensions always admit a $\mathrm{U}(1)$ breaking distortion, leading to a distortion of AdS_5 at asymptotic infinity [17, 18]. Closed timelike curves (CTC's) may be avoided in these Gödel-like backgrounds, provided the distortion is sufficiently small.

This paper is organized as follows. In section 2, we review gauged $\mathcal{N} = 8$ supergravity in five dimensions, and discuss its truncation to the STU model coupled to three incomplete hypermultiplets. In section 3, we discuss the G-structure approach to constructing supersymmetric backgrounds. In section 4, we present the system of first-order equations for supersymmetric backgrounds that preserve a time-like Killing vector and which have a bi-axial four-dimensional Kähler base space. In section 5, we present some explicit solutions which do not involve hyperscalars, such as black holes, solitons and time machines. These solutions can be generalized by relaxing the $\mathrm{SU}(2)_L \times \mathrm{U}(1)$ isometry to $\mathrm{SU}(2)_L$. In section 6, we discuss solutions which do involve hyperscalars, and which are generalizations of the AdS bubbles [10]. We discuss bubbling generalizations of the Klemm-Sabra black holes in section 7, and conclude in section 8. Details regarding differential identities for the invariant tensors, as well as the system of equations governing a tri-axial four-dimensional Kähler base space, are left for the appendices.

2. Truncation of $\mathcal{N} = 8$ supergravity

Since we are interested in truncating $\mathcal{N} = 8$ supergravity into either matter coupled $\mathcal{N} = 2$ supergravity or bosonic subsectors thereof, we begin with the decomposition of the $\mathcal{N} = 8$

$\mathcal{N}=2$ multiplet	fields	E_0 values	$\mathrm{SU}(3)\times\mathrm{U}(1)$
graviton	$(h_{\mu\nu},\psi_\mu,A_\mu)$	$(4,\frac{7}{2},3)$	$(\mathbf{1}_0,\mathbf{1}_{\pm1},\mathbf{1}_0)$
gravitino	$(\psi_\mu,A_\mu,B_{\mu\nu},\lambda)$	$(\frac{7}{2},3,3,\frac{5}{2})$	$(\mathbf{3}_0,\mathbf{3}_1,\mathbf{3}_{-1},\mathbf{3}_0)_{1/3}$+conj.
vector	(A_μ,λ,ϕ)	$(3,\frac{5}{2},2)$	$(\mathbf{8}_0,\mathbf{8}_{\pm1},\mathbf{8}_0)$
tensor	$(\lambda,B_{\mu\nu},\phi,\lambda)$	$(\frac{7}{2},3,3,\frac{5}{2})$	$(\mathbf{3}_0,\mathbf{3}_{-1},\mathbf{3}_{-1},\mathbf{3}_{-2})_{1/3}$+conj.
hypermatter (1)	(ϕ,λ,ϕ)	$(3,\frac{5}{2},2)$	$(\mathbf{6}_0,\mathbf{6}_{-1},\mathbf{6}_{-2})_{2/3}$+conj.
hypermatter (2)	(ϕ,λ,ϕ)	$(4,\frac{7}{2},3)$	$(\mathbf{1}_0,\mathbf{1}_{-1},\mathbf{1}_{-2})$+conj.

Table 1: Decomposition of the $\mathcal{N}=8$ supergravity multiplet into $\mathcal{N}=2$ multiplets under $\mathrm{SU}(4)\supset$ $\mathrm{SU}(3)\times\mathrm{U}(1)$.

supergravity multiplet into $\mathcal{N}=2$ multiplets. This is presented in table 1, where we also give the lowest weight energies E_0 and the representations under $\mathrm{SU}(3)\times\mathrm{U}(1)\subset\mathrm{SU}(4)$. Here $\mathrm{U}(1)$ is the R-symmetry of the $\mathcal{N}=2$ theory embedded within the $\mathrm{SO}(6)\simeq\mathrm{SU}(4)$ R-symmetry of the full $\mathcal{N}=8$ theory.

Note that the standard truncation of $\mathcal{N}=8$ to the STU model ($\mathcal{N}=2$ supergravity coupled to two vector multiplets) corresponds to retaining two of the eight vectors in the maximal torus of $\mathrm{SU}(3)$. In addition to gravity, the STU model has three abelian vectors A_μ^I (one of which is the graviphoton) and two unconstrained scalars, which may be traded off for three scalars X^I satisfying the cubic constraint $X^1X^2X^3=1$. Since this is a model with vector multiplets, it is naturally described using very special geometry.

In addition to the matter content of the STU model, we are interested in retaining three additional scalars φ_I of the $\mathcal{N}=8$ theory. From the $\mathcal{N}=8$ point of view, these additional scalars share a common origin with the X^I scalars as the diagonal elements of the $\mathrm{SL}(6,\mathbb{R})/\mathrm{SO}(6)$ coset representative

$$\mathcal{M}=\mathrm{diag}(\sqrt{X^1}e^{\varphi_1/2},\sqrt{X^1}e^{-\varphi_1/2},\sqrt{X^2}e^{\varphi_2/2},\sqrt{X^2}e^{-\varphi_2/2},\sqrt{X^3}e^{\varphi_3/2},\sqrt{X^3}e^{-\varphi_3/2}),\quad(2.1)$$

which is contained inside the $E_{6(6)}/USp(8)$ scalar manifold of $\mathcal{N}=8$ supergravity. However, despite this common origin, the φ_I scalars fall outside of the $\mathcal{N}=2$ vector multiplets. In particular, these additional scalars are parts of hypermultiplets of the first type listed in table 1. While, it is clear that they alone are insufficient to comprise the bosonic parts of complete multiplets in themselves, the supersymmetry analysis below nevertheless allows us to obtain solutions to the full $\mathcal{N}=8$ theory in which only this restricted set of fields is active.

In principle, the addition of hypermatter requires us to consider the full matter coupled $\mathcal{N}=2$ gauged supergravity [19]. However, for simplicity, we restrict ourselves to the STU model coupled to the three additional φ_I scalars. As a result, we shall not need the entire machinery of $\mathcal{N}=2$ matter couplings (i.e., very special geometry for vector multiplets and quaternionic geometry for hypermultiplets), but will instead follow a direct reduction of the $\mathcal{N}=8$ expressions into their $\mathcal{N}=2$ counterparts. We thus begin with a review of the $\mathcal{N}=8$ theory, which serves as the initial point of our analysis.

2.1 The $\mathcal{N} = 8$ supergravity

Gauged $\mathcal{N} = 8$ supergravity in five dimensions was constructed in [20–22]. The bosonic fields consist of the metric $g_{\mu\nu}$, SO(8) adjoint gauge fields $A_{\mu\,IJ}$, antisymmetric tensors $B_{\mu\nu}{}^{I\,\alpha}$ transforming as $(\mathbf{6}, \mathbf{2})$ under SO(6) × SL(2, $\mathbb{R}$) and 42 scalars $V_{AB}{}^{ab}$ parameterizing the coset $E_{6(6)}/USp(8)$ and transforming as $\mathbf{20'} + \mathbf{10} + \overline{\mathbf{10}} + \mathbf{1} + \mathbf{1}$ under SO(6). The fermions are the 8 gravitini $\psi_{\mu\,a}$ and 48 dilatini χ_{abc}, all transforming under $USp(8)$.

Following the notation of [21, 22], but working in signature $(-, +, +, +, +)$, the gauged $\mathcal{N} = 8$ Lagrangian has the form

$$e^{-1}\mathcal{L} = R - \frac{1}{6}P_{\mu\,abcd}P^{\mu\,abcd} - \frac{1}{8}H_{\mu\nu\,ab}H^{\mu\nu\,ab} + \frac{1}{2}\overline{\psi}_\mu{}^a \gamma^{\nu\nu\rho}D_\rho\psi_{\rho\,a} + \frac{1}{12}\overline{\chi}^{abc}\gamma^\mu D_\mu\chi_{abc} - V + \cdots, \tag{2.2}$$

where we have only written the kinetic terms explicitly. Here D_μ is the gravitational as well as SL(6, $\mathbb{R}$) × $USp(8)$ covariant derivative, and

$$P_\mu{}^{abcd} = \tilde{V}^{ab\,AB}D_\mu V_{AB}{}^{cd}, \qquad P_\mu{}^{abcd} \equiv P_\mu{}^{[abcd]|}, \tag{2.3}$$

is the scalar kinetic term. The two-forms $H_{\mu\nu\,ab}$ are a combination of the gauge fields and anti-symmetric tensors

$$H_{\mu\nu}{}^{ab} = F_{\mu\nu\,IJ}V^{IJ\,ab} + B_{\mu\nu}{}^{I\alpha}V_{I\alpha}{}^{ab}. \tag{2.4}$$

Finally, the scalar potential V may be written in terms of the W-tensor as

$$V = -\frac{1}{2}g^2(2W_{ab}^2 - W_{abcd}{}^2), \qquad W_{abcd} = \epsilon^{\alpha\beta}\delta^{IJ}V_{I\alpha\,ab}V_{J\beta\,cd}. \tag{2.5}$$

To leading order, the supersymmetry transformations for the gravitini and dilatini take the form

$$\delta\psi_{\mu\,a} = D_\mu\epsilon_a + \frac{i}{12}(\gamma_\mu{}^{\nu\rho} - 4\delta_\mu^\nu\gamma^\rho)F_{\nu\rho\,ab}\epsilon^b - \frac{i}{3}g\gamma_\mu W^c{}_{acb}\epsilon^b,$$

$$\delta\chi_{abc} = -i\sqrt{2}\gamma^\mu P_{\mu\,abcd}\epsilon^d + \frac{3}{4\sqrt{2}}\gamma^{\mu\nu}F_{\mu\nu\,[ab}\epsilon_{c]|} + 3\sqrt{2}gW_{d[abc]|}\epsilon^d. \tag{2.6}$$

Note that the $USp(8)$ indices $a, b, \ldots$ are raised and lowered with the symplectic matrix Ω_{ab}, and the symplectic-Majorana Weyl spinors satisfy

$$\overline{\lambda}^a \equiv \lambda_a^\dagger\gamma^0 = \Omega^{ab}\lambda_b^T C, \tag{2.7}$$

where C is the charge conjugation matrix.

Before considering the truncation to $\mathcal{N} = 2$, however, we first examine the scalar sector of the theory. Although the complete $\mathcal{N} = 8$ scalar manifold is given by the coset $E_{6(6)}/USp(8)$, the gauging of SO(6) $\subset USp(8)$ complicates the explicit treatment of these scalars. For this reason, we now consider the simpler subsector of the scalar manifold corresponding to taking SL(6, $\mathbb{R}$) × SL(2, $\mathbb{R}$) $\subset E_{6(6)}$. Furthermore, this subset of scalars has a natural Kaluza-Klein origin from the S^5 reduction of IIB supergravity; the $\mathbf{20'}$ scalars living on SL(6, $\mathbb{R}$)/ SO(6) correspond to metric deformations on S^5, while the SL(2, $\mathbb{R}$)/ SO(2)

scalars descend directly from the ten-dimensional IIB dilaton-axion. In particular, these $\mathbf{20'}$ scalars, along with the SO(8) gauge fields, were precisely the fields retained in the S^5 Pauli reduction of [23]. Note that, while this system is a consistent bosonic truncation of $\mathcal{N} = 8$ supergravity, it is however not supersymmetric (even if fermions were to be included). This is because the $\mathbf{20'}$ scalars, corresponding to $E_0 = 2$ in table 1, comprise only a subset of the first hypermultiplet listed. The remaining scalars in the hypermultiplet originate from the reduction of the complexified three-form in IIB on S^5.

Denoting the $\mathrm{SL}(6, \mathbb{R})/\mathrm{SO}(6)$ and $\mathrm{SL}(2, \mathbb{R})/\mathrm{SO}(2)$ coset representatives by $\mathcal{M}^I{}_J$ and $\mathcal{N}^\alpha{}_\beta$, respectively, we follow [22] and obtain the $E_{6(6)}$ elements

$$U^{MN}{}_{IJ} = 2\mathcal{M}^{-1\,[M}{}_{[I}\mathcal{M}^{-1\,N]}{}_{J]}, \qquad U_{J\beta}{}^{I\alpha} = \mathcal{M}^I{}_J\mathcal{N}^\alpha{}_\beta. \tag{2.8}$$

Transforming to a $USp(8)$ basis using a set of imaginary antisymmetric SO(7) Dirac matrices Γ_i ($i = 0, 1, \ldots, 6$ while $I = 1, \ldots, 6$) results in the coset representatives

$$V^{IJ\,ab} = \frac{1}{4}(\Gamma_{KL})^{ab}\mathcal{M}^{-1\,I}{}_K\mathcal{M}_L^{-1\,J}, \qquad V_{I\alpha}{}^{ab} = \frac{1}{2\sqrt{2}}(\Gamma_{K\beta})^{ab}\mathcal{M}^K{}_I\mathcal{N}^\beta{}_\alpha, \tag{2.9}$$

along with the inverses

$$\tilde{V}_{IJ\,ab} = \frac{1}{2}(\Gamma_{KL})^{ab}\mathcal{M}^K{}_J\mathcal{M}^L{}_J, \qquad \tilde{V}_{ab}{}^{I\alpha} = -\frac{1}{2\sqrt{2}}(\Gamma_{K\beta})^{ab}\mathcal{M}^{-1\,I}{}_K\mathcal{N}^{-1\,\alpha}{}_\beta. \tag{2.10}$$

In this case, the W-tensor of (2.5) reduces to

$$W_{abcd} = \frac{i}{8}[(\Gamma_I)_{ab}(\Gamma_J\Gamma_0)_{cd} - (\Gamma_I\Gamma_0)_{ab}(\Gamma_J)_{cd}]M^{IJ}, \tag{2.11}$$

where

$$M^{IJ} = \mathcal{M}^I{}_K\mathcal{M}^J{}_L\delta^{KL}, \quad \text{or} \quad M = \mathcal{M}\mathcal{M}^T. \tag{2.12}$$

Using

$$W_{ab} = -\frac{1}{4}\delta_{ab}\mathrm{Tr}M, \qquad (W_{abcd})^2 = 2\mathrm{Tr}(M^2), \tag{2.13}$$

and substituting into (2.5) yields the scalar potential

$$V = -\frac{1}{2}g^2[(\mathrm{Tr}M)^2 - 2\mathrm{Tr}(M^2)]. \tag{2.14}$$

Note that the $\mathrm{SL}(2, \mathbb{R})$ scalars (or equivalently the IIB dilaton-axion) do not enter the potential.

Continuing with this specialization of the scalar sector, we find that the gauge fields enter in the combination

$$F_{\mu\nu}{}^{ab} = F_{\mu\nu\,IJ}V^{IJ\,ab} = \frac{1}{4}F_{\mu\nu\,IJ}(\Gamma_{KL})^{ab}\mathcal{M}^{-1\,I}{}_K\mathcal{M}^{-1\,J}{}_L. \tag{2.15}$$

The final quantity we need is the scalar kinetic term $P_\mu{}^{abcd}$ defined in (2.3). The condition that $P_\mu{}^{abcd}$ is automatically symplectic-trace free determines the composite $USp(8)$ connection $Q_{\mu\,a}{}^b$ to be

$$Q_{\mu\,a}{}^b = \frac{1}{2}(\Gamma_{IJ})^{ab}(\mathcal{M}\partial_\mu\mathcal{M}^{-1})^I{}_J + \frac{i}{2}(\Gamma_0)^{ab}\epsilon_{\alpha\beta}(\mathcal{N}\partial_\mu\mathcal{N}^{-1})^\alpha{}_\beta - \frac{1}{4}gA_{\mu\,IJ}(\Gamma_{KL})^{ab}\mathcal{M}^{-1\,I}{}_K\mathcal{M}^L{}_J. \tag{2.16}$$

This shows up both in the covariant derivative in the gravitino variation and in the scalar kinetic term

$$
\begin{aligned}
P_\mu{}^{abcd} &= \tilde{V}^{ab}{}_{IJ} D_\mu V^{IJ\,cd} + 2\tilde{V}^{ab\,I\alpha} D_\mu V_{I\alpha}{}^{cd}, \\
&= \tilde{V}^{ab}{}_{IJ}[\partial_\mu V^{IJ\,cd} - Q_{\mu\,e}{}^c V^{IJ\,ed} - Q_{\mu\,e}{}^d V^{IJ\,ce} - 2gA_{\mu\,IK} V^{KJ\,cd}] \\
&\quad + 2\tilde{V}^{ab\,I\alpha}[\partial_\mu V_{I\alpha}{}^{cd} - Q_{\mu\,e}{}^c V_{I\alpha}{}^{ed} - Q_{\mu\,e}{}^d V_{I\alpha}{}^{ce} - gA_{\mu\,IJ} V_{J\alpha}{}^{cd}] \\
&= \frac{1}{4}(\Gamma_{I\beta})^{ab}(\Gamma_{I\alpha})^{cd}(\mathcal{N}\partial_\mu \mathcal{N}^{-1})^\alpha{}_\beta \\
&\quad + \frac{1}{4}[(\Gamma_{IM})^{ab}(\Gamma_{JM})^{cd} + (\Gamma_J)^{ab}(\Gamma_I)^{cd} - (\Gamma_J\Gamma_0)^{ab}(\Gamma_I\Gamma_0)^{cd}](\mathcal{M}\partial_\mu \mathcal{M}^{-1})^I{}_J \\
&\quad - \frac{1}{4}gA_{\mu\,IJ}[(\Gamma_{KM})^{ab}(\Gamma_{LM})^{cd} - (\Gamma_L)^{ab}(\Gamma_K)^{cd} + (\Gamma_L\Gamma_0)^{ab}(\Gamma_K\Gamma_0)^{cd}]\mathcal{M}^K{}_I \mathcal{M}^{-1\,J}{}_L \\
&\quad - [Q_\mu^{ac}\Omega^{bd} + Q_\mu^{bd}\Omega^{ac} - Q_\mu^{ad}\Omega^{bc} - Q_\mu^{bc}\Omega^{ad} - \frac{1}{2}Q_\mu^{cd}\Omega^{ab}].
\end{aligned}
\tag{2.17}
$$

These expressions, in principle, allow us to work out the full gravitino and dilatino variations (2.6) in terms of the explicit parameterization (2.8) of the $\mathrm{SL}(6,\mathbb{R})\times\mathrm{SL}(2,\mathbb{R})$ scalars.

2.2 The truncation to $\mathcal{N}=2$

As indicated in [22], the gauged $\mathcal{N}=8$ theory admits two maximal truncations to $\mathcal{N}=2$ supergravity. The first retains only the hypermatter shown in table 1 coupled to the $\mathcal{N}=2$ graviton multiplet, while the second corresponds to keeping only the vector and tensor multiplets. A further consistent truncation of this second case to the zero weight sector of SU(3) then yields the standard STU model, namely $\mathcal{N}=2$ supergravity coupled to two vector multiplets.

We are mainly interested in a truncation of the above $\mathcal{N}=8$ theory, where we retain the three gauge fields

$$
A^1 = A^{12}, \qquad A^2 = A^{34}, \qquad A^3 = A^{56},
\tag{2.18}
$$

on the maximal torus of SO(6), along with the five scalars (2.1) parameterizing the diagonal component of the $\mathrm{SL}(6;\mathbb{R})/\mathrm{SO}(6)$ coset. Using (2.12), we have

$$
M = \mathrm{diag}(X^1 e^{\varphi_1}, X^1 e^{-\varphi_1}, X^2 e^{\varphi_2}, X^2 e^{-\varphi_2}, X^3 e^{\varphi_3}, X^3 e^{-\varphi_3}),
\tag{2.19}
$$

in which case $\mathrm{Tr}\,M = 2\sum_I X^I \cosh\varphi_I$ and $\mathrm{Tr}\,M^2 = 2\sum_I (X^I)^2(\cosh^2\varphi_I + \sinh^2\varphi_I)$. As a result, from (2.14) we obtain the scalar potential

$$
V = 2g^2\left(\sum_I (X^I)^2 \sinh^2\varphi_I - 2\sum_{I<J} X^I X^J \cosh\varphi_I \cosh\varphi_J\right).
\tag{2.20}
$$

Note that this may be derived from a superpotential

$$
W = g\sum_I X^I \cosh\varphi_I,
\tag{2.21}
$$

using the relation

$$
V = 2\sum_\alpha (\partial_\alpha W)^2 - \frac{4}{3}W^2,
\tag{2.22}
$$

where $\alpha = 1, 2, \ldots, 5$ runs over the five unconstrained scalars.

After some manipulation of the scalar kinetic term (2.17), we find that the truncated bosonic action is

$$
\begin{aligned}
e^{-1}\mathcal{L} = {} & R - \frac{1}{2}\partial\phi_\alpha^2 - \frac{1}{2}\partial\varphi_I^2 - \frac{1}{4}(X^I)^{-2}(F_{\mu\nu}^I)^2 - 2g^2\sinh^2\varphi_I(A_\mu^I)^2 \\
& -V - \frac{1}{4}\epsilon^{\mu\nu\rho\lambda\sigma}F_{\mu\nu}^1 F_{\rho\lambda}^2 A_\sigma^3.
\end{aligned}
\tag{2.23}
$$

Note that the term proportional to $(A_\mu^I)^2$ originates from the SO(6) gauging in (2.16) and (2.17). The lack of manifest gauge invariance in this action is a consequence of the truncation to incomplete hypermultiplets.

In addition, the $\mathcal{N} = 8$ supersymmetry transformations decompose into four sets, each corresponding to a different embedding of $\mathcal{N} = 2$ into $\mathcal{N} = 8$. From a particular $\mathcal{N} = 2$ perspective, we may focus on a single set. However, note that in general the other three sets of supersymmetries may be completely broken, unless additional symmetries are present beyond what is imposed by the $\mathcal{N} = 2$ analysis below. For example, three-charge non-rotating solutions preserve $1/2$ of the $\mathcal{N} = 2$ supersymmetries, but only $1/8$ of the $\mathcal{N} = 8$ ones (corresponding to preserving four real supercharges in either case).

We end up with the $\mathcal{N} = 2$ sector supersymmetry transformations

$$
\begin{aligned}
\delta\psi_{\mu i} = {} & \nabla_\mu\epsilon_i + \frac{i}{24}(\gamma_\mu{}^{\nu\rho} - 4\delta_\mu^\nu\gamma^\rho)\mathcal{F}_{\nu\rho}\epsilon_i + \frac{1}{2}g\mathcal{A}_\mu\epsilon_{ij}\epsilon_j + \frac{i}{6}W\gamma_\mu\epsilon_{ij}\epsilon_j, \\
\delta\lambda_{I i} = {} & -i\gamma^\mu\partial_\mu\varphi_I\epsilon_i + 2ig\gamma^\mu A_\mu^I\sinh\varphi_I\epsilon_{ij}\epsilon_j - 2gX^I\sinh\varphi_I\epsilon_{ij}\epsilon_j, \\
\delta\chi_i^{(1)} = {} & -i\gamma^\mu\partial_\mu\log((X^1)^2/(X^2 X^3))\epsilon_i - \frac{1}{2}\gamma^{\mu\nu}(2(X^1)^{-1}F_{\mu\nu}^1 - (X^2)^{-1}F_{\mu\nu}^2 - (X^3)^{-1}F_{\mu\nu}^3)\epsilon_i \\
& -2g(2X^1\cosh\varphi_1 - X^2\cosh\varphi_2 - X^3\cosh\varphi_3)\epsilon_{ij}\epsilon_j, \\
\delta\chi_i^{(2)} = {} & -i\gamma^\mu\partial_\mu\log((X^2)^2/(X^1 X^3))\epsilon_i - \frac{1}{2}\gamma^{\mu\nu}(-(X^1)^{-1}F_{\mu\nu}^1 + 2(X^2)^{-1}F_{\mu\nu}^2 - (X^3)^{-1}F_{\mu\nu}^3)\epsilon_i \\
& -2g(-X^1\cosh\varphi_1 + 2X^2\cosh\varphi_2 - X^3\cosh\varphi_3)\epsilon_{ij}\epsilon_j,
\end{aligned}
\tag{2.24}
$$

where we have defined the graviphoton combinations

$$
\begin{aligned}
\mathcal{A}_\mu &\equiv A_\mu^1\cosh\varphi_1 + A_\mu^2\cosh\varphi_2 + A_\mu^3\cosh\varphi_3, \\
\mathcal{F}_{\mu\nu} &\equiv (X^1)^{-1}F_{\mu\nu}^1 + (X^2)^{-1}F_{\mu\nu}^2 + (X^3)^{-1}F_{\mu\nu}^3,
\end{aligned}
\tag{2.25}
$$

and where the superpotential W is given in (2.21). The spinors ϵ_i, $i = 1, 2$ are now to be considered as $\mathcal{N} = 2$ spinors.

The gravitino and gaugino variations can almost be written in very special geometry language (for the STU model) where, instead of taking $V_I = 1/3$, we use $V_I = \frac{1}{3}\cosh\varphi_I$. The φ_I scalars are parts of hypermultiplets and, when frozen to their constant values $\varphi_I = 0$, the gauging parameters V_I take on their standard constant values.

Note also that in the ungauged theory (obtained by taking $g \to 0$), the hypermultiplets decouple from the vector multiplets, at least in the supersymmetry transformations. This is just the standard decoupling of $\mathcal{N} = 2$ vector and hyper multiplets. Furthermore, in the truncation to the dilatonic hypermultiplet scalars φ_I, they also decouple from the gravitino multiplet. (The axionic ones will show up via the composite connection $Q_{\mu a}{}^b$.)

3. Supersymmetry analysis

We shall use the invariant tensor approach for constructing supersymmetric backgrounds. This G-structure analysis has been successfully applied to many systems, including minimal $\mathcal{N} = 2$ supergravity in four dimensions [12, 13] as well as minimal ungauged [14] and gauged [15] $\mathcal{N} = 2$ supergravities in five dimensions. The inclusion of vectors in the five-dimensional gauged $\mathcal{N} = 2$ case was investigated in [5, 6] in the context of constructing supersymmetric black holes.

We are of course interested in constructing supersymmetric backgrounds where the hypermatter scalars φ_I are active. In this context, the BPS conditions for obtaining static spherically symmetric solutions were analyzed in [24] for gauged $\mathcal{N} = 2$ supergravity coupled to hypermatter. This was further generalized in [25, 26] for the complete system including both vector and hypermultiplets. (See also [27] for a complete analysis of ungauged supergravity coupled to hypermatter.) These studies, however, assumed spherical symmetry from the outset, an assumption that we wish to relax. Thus, we shall mainly follow the invariant tensor procedure of [14, 15, 5, 6]. This procedure starts with a construction of all tensors formed as bilinears of the Killing spinor ϵ_i followed by an examination of algebraic and differential identities related to these tensors, which we now consider.

3.1 Spinor bilinear identities

Note that ϵ_i is an $\mathcal{N} = 2$ symplectic-Majorana spinor, with i an $Sp(2) \simeq \mathrm{SU}(2)$ index. In particular, it carries eight real spinor components. We may form a complete set of real bilinears

$$f = \frac{i}{2}\bar{\epsilon}^i \epsilon_i, \qquad K_\mu = \frac{1}{2}\bar{\epsilon}^i \gamma_\mu \epsilon_i, \qquad \Phi^a_{\mu\nu} = \frac{1}{2}\bar{\epsilon}^i (\tau^a)_{ij} \gamma_\mu \epsilon_j, \tag{3.1}$$

where τ^a are the usual Pauli matrices. We take as a convention $\frac{1}{5!}\epsilon_{\mu\nu\rho\lambda\sigma}\gamma^{\mu\nu\rho\lambda\sigma} = i$ along with $\epsilon_{01234} = 1$.

The standard Fierz identities give the normalization relations

$$K^2 = -f^2, \qquad (\Phi^a_{\mu\nu})^2 = 12f^2, \tag{3.2}$$

along with

$$\begin{aligned}
i_K \Phi^a &= 0, \\
i_K * \Phi^a &= -f\Phi^a, \\
\Phi^a \wedge \Phi^b &= -2\delta^{ab} f * K, \\
\Phi^a_{\lambda\mu}\Phi^{b\,\lambda}{}_\nu &= \delta^{ab}(f^2 g_{\mu\nu} + K_\mu K_\nu) - \epsilon^{abc} f \Phi^c_{\mu\nu},
\end{aligned} \tag{3.3}$$

where for any p-form ω we define $(i_K\omega)_{\mu_1\cdots\mu_{p-1}} = K^\nu \omega_{\nu\mu_1\cdots\mu_{p-1}}$. These identities indicate that the set (K, Φ^2) defines a preferred $\mathrm{U}(2)$ structure. In the ungauged case, the addition of Φ^1 and Φ^3 would yield a preferred $\mathrm{SU}(2)$ structure. However, here they are charged under the gauged $\mathrm{U}(1)$, and hence are only covariant and not invariant.

Integrability of the $\mathrm{U}(2)$ structure may be investigated through the differential identities which arise from the supersymmetry variations. These are presented in appendix A.

As usual, symmetrization of the $\nabla_\mu K_\nu$ identity arising from the gravitino variation (A.1) demonstrates that K^μ is a Killing vector:

$$2\nabla_{(\mu} K_{\nu)} = 0. \tag{3.4}$$

This, combined with (3.2), ensures that K^μ is an everywhere non-spacelike Killing vector. Since we are interested in constructing black holes (and related solitonic bubbles), we take the timelike case where

$$K^2 = -f^2 < 0. \tag{3.5}$$

3.2 Specializing the metric

We now assume K^μ is a timelike Killing vector with norm $K^2 = -f^2$ where $f \neq 0$. For simplicity of notation, we take $f > 0$. (The $f < 0$ case is similar, and involves a modified choice of signs. However, it does not give rise to any intrinsically new solutions.) In this case, we may specialize the metric to be of the form

$$ds^2 = -f^2(dt + \omega)^2 + f^{-1} h_{mn} dx^m dx^n. \tag{3.6}$$

Note that we take $K = \partial/\partial t$, so that $K = -f^2(dt + \omega) = -f e^0$ where $e^0 = f(dt + \omega)$.

Given that $i_K \Phi^a = 0$ from (3.3), we see that the two-forms Φ^a live on the four-dimensional base with metric h_{mn}. The remaining identities in (3.3) are then equivalent to

$$*_4 \Phi^a = -\Phi^a, \qquad \Phi^a \wedge \Phi^b = -2\delta^{ab} *_4 1, \qquad \Phi^a_{mp} \Phi^b_{nq} h^{pq} = \delta^{ab} h_{mn} - \epsilon^{abc} \Phi^c_{mn}. \tag{3.7}$$

This indicates that the three Φ^a form a set of anti-self-dual 2-forms on the base that satisfy the algebra of unit quaternions. In the ungauged case, this is sufficient to demonstrate a preferred SU(2) structure; here Φ^2 defines a U(2) structure, while Φ^1 and Φ^3 are charged under the gauged U(1).

To make the structure explicit, we define the canonical 2-form J along with a complex 2-form Ω according to

$$J = \Phi^2, \qquad \Omega = \Phi^1 + i\Phi^3. \tag{3.8}$$

This set (J, Ω) determines the U(2) structure on the base,

$$J \wedge \Omega = 0, \qquad J \wedge J = \frac{1}{2} \Omega \wedge \Omega^* = -2 *_4 1. \tag{3.9}$$

Integrability of J and Ω will be taken up below, when we consider the differential identities.

3.3 Determining the gauge fields

In order to obtain a supersymmetric background, we need to determine not only the metric $g_{\mu\nu}$ (or equivalently the quantities f, ω and h_{mn}) but also the matter fields A^I, X^I and φ_I. We begin with the gauge fields. Firstly, using (A.3), which we take as either a gauge condition (when $g \sinh \varphi_I = 0$) or as a consequence of the hyperino transformations, we may write the potentials as

$$A^I = f^{-1} X^I K + \beta^I = -X^I e^0 + \beta^I, \tag{3.10}$$

where β^I lives exclusively on the base (i.e., $i_K \beta^I = 0$). The field strengths are then

$$F^I = dA^I = -d(X^I e^0) + d\beta^I = f^{-2}d(fX^I) \wedge K - fX^I d\omega + d\beta^I. \qquad (3.11)$$

Note that the one-form identity (A.7) is automatically satisfied.

To proceed, we may turn to the two-form identities (A.8). For simplicity, we define the components $\overline{F}^I$ of the field strengths on the base by writing (3.11) as

$$F^I = f^{-2}d(fX^I) \wedge K + \overline{F}^I. \qquad (3.12)$$

The two-form identities then reduce to

$$*_4\left((X^I)^{-1}\overline{F}^I\right) + (X^J)^{-1}\overline{F}^J + (X^K)^{-1}\overline{F}^K = -fd\omega + 2gf^{-1}JX^I \cosh\varphi_I, \qquad (3.13)$$

where $I \neq J \neq K$. By breaking this up into self-dual and anti-self dual parts, we obtain a complete determination of the anti-self dual components

$$\left((X^I)^{-1}\overline{F}^I\right)^- = -f(d\omega)^- + gf^{-1}J(X^J \cosh\varphi_J + X^K \cosh\varphi_K) \qquad (3.14)$$

and a single condition on the sum of the self-dual components

$$\left((X^1)^{-1}\overline{F}^1\right)^+ + \left((X^2)^{-1}\overline{F}^2\right)^+ + \left((X^3)^{-1}\overline{F}^3\right)^+ = -f(d\omega)^+. \qquad (3.15)$$

In terms of $d\beta^I$, these conditions become

$$(d\beta^I)^- = gf^{-1}J\left(\frac{1}{X^J}\cosh\varphi_K + \frac{1}{X^K}\cosh\varphi_J\right), \qquad (3.16)$$

and

$$\frac{1}{X^1}(d\beta^1)^+ + \frac{1}{X^2}(d\beta^2)^+ + \frac{1}{X^3}(d\beta^3)^+ = 2f(d\omega)^+. \qquad (3.17)$$

To show that the base metric h_{mn} is Kähler, we note from the first equation of (A.9) that $dJ = 0$ is trivially satisfied. In order to examine $d\Omega$, we decompose the graviphoton $\mathcal{A}$ defined in (2.25) into timelike and spatial components using (3.10). If we multiply (A.3) by $\cosh\varphi_I$ and sum over I, we see that the graviphoton necessarily satisfies the condition

$$gi_K\mathcal{A} = -fW. \qquad (3.18)$$

This ensures that the timelike component of $\mathcal{A}$ cancels against the superpotential term in (A.9), leaving

$$d\Omega = -ig(\beta^1 \cosh\varphi_1 + \beta^2 \cosh\varphi_2 + \beta^3 \cosh\varphi_3) \wedge \Omega. \qquad (3.19)$$

Combined with $dJ = 0$, we see that the base is indeed Kähler,[2] with Ricci form satisfying

$$\mathcal{R} = -g\, d(\beta^1 \cosh\varphi_1 + \beta^2 \cosh\varphi_2 + \beta^3 \cosh\varphi_3). \qquad (3.20)$$

---

[2]The conditions for Kählerity can be expressed as $dJ = 0$, $J \wedge \Omega = 0$, $d\Omega = i\omega \wedge \Omega$ for some 1-form ω. The 1-form ω is arbitrary up to the addition of any $(0, 1)$-form. There exists a choice for ω such that $d\omega = \mathcal{R}$, the Ricci form.

It is now easy to see that the remaining 0-form gaugino identities in (A.6) are satisfied. Furthermore, with some work, we may also verify that the additional 3-form identities (A.9) are satisfied as well. Note, in particular, that the identities related to J (i.e., the $a = 2$ identities) require that the graviphoton-free combinations of $\overline{F}^I$ be $(1,1)$-forms on the base

$$J_{[m}{}^p \overline{F}^{(\alpha)}_{n]p} = 0. \tag{3.21}$$

This is trivially satisfied because the self-dual part of $\overline{F}^{(\alpha)}$ is automatically $(1,1)$, while from (3.14) we see that the graviphoton-free anti-self-dual part is proportional to J, which is itself a $(1,1)$-form. We have not explicitly checked the 4-form identities (A.10), but expect them to hold without any new conditions.

3.4 Determining the hypermatter scalars

So far, other than using (A.3) to determine the time component of A^I, we have not focused on the hypermatter scalars φ_I. Thus, the above analysis is essentially identical to that of [15, 5, 6] for minimal gauged $\mathcal{N} = 2$ supergravity and gauged $\mathcal{N} = 2$ supergravity coupled to vector multiplets. However, we now turn to the hyperino identities (A.11). The zero-form identities have already been accounted for, so we proceed directly with the 1-form identity, which requires that φ_I live on the four-dimensional base, and satisfy

$$d\varphi_I = -2g \sinh \varphi_I J_m{}^n \beta_n^I dx^m. \tag{3.22}$$

This relates the hypermultiplet scalars φ_I with the spatial components of the gauge fields β^I. Note that this can equivalently be written as

$$J \wedge d\varphi_I = 2g \sinh \varphi_I *_4 \beta^I. \tag{3.23}$$

As it turns out, this condition is sufficient to ensure that all the remaining hyperino identities are satisfied. To see this, we may turn directly to the supersymmetry transformation $\delta\lambda_{Ii}$ given in (2.24). Substituting in (3.22) as well as the gauge field decomposition (3.10) gives

$$\delta\lambda_{Ii} = 2ig \sinh \varphi_I \beta_n \gamma^m \epsilon_{ij} [\delta_m^n \delta_j^k - J_m{}^n \epsilon_{jk}]\epsilon_k - 2g X^I \sinh \varphi_I \epsilon_{ij}[1 + i\gamma^0]\epsilon_j. \tag{3.24}$$

This expression must vanish in order for ϵ_i to be a Killing spinor. So long as $g \sinh \varphi_I \neq 0$, the second term in (3.24) yields the familiar condition

$$i\gamma^0 \epsilon_i = -\epsilon_i. \tag{3.25}$$

If this were the only condition, then the solution would be 1/2 BPS. However, we must also ensure the vanishing of the first term in (3.24). This may be accomplished by noting that, so long as β_n is generic, we must demand

$$\gamma^m [\delta_m^n \delta_i^j - J_m{}^n \epsilon_{ij}]\epsilon_j = 0. \tag{3.26}$$

Multiplying on the left by $\frac{1}{4}\gamma_n$ then gives

$$\left[\delta_i^j + \frac{1}{4}(J \cdot \gamma)\epsilon_{ij}\right]\epsilon_j = 0. \tag{3.27}$$

Since $(J \cdot \gamma)\epsilon_{ij}$ has eigenvalues $\pm 4, 0, 0$, we see that this yields a 1/4 BPS projection. Furthermore, since J is anti-self dual:

$$[(J \cdot \gamma)i\tau^2]^2 = 8(1 + \gamma^{1234}) = 8(1 - i\gamma^0) \tag{3.28}$$

we see that the projection (3.27) is compatible with (3.25), and hence the complete system remains 1/4 BPS when both projections inherent in (3.24) are taken into account.

3.5 Completing the solution

To complete the solution, we must impose the F^I equations of motion. Note that by making the ansatz (3.10) on the gauge potential, we are guaranteed to satisfy the Bianchi identities. From (2.23), the F^I equation of motion reads

$$d\left(*\frac{1}{(X^I)^2}F^I\right) = F^J \wedge F^K - 4g^2 \sinh^2 \varphi_I * A^I. \tag{3.29}$$

Using the explicit forms for A^I and F^I given in (3.10) and (3.11), we see that this equation decomposes into one whose component lies along e^0, and one which only resides on the base. The former turns out to be trivially satisfied, provided the supersymmetry conditions (3.16), (3.17) and (3.23) hold. On the other hand, the part of (3.29) which lies on the base gives rise to the second-order equation

$$d *_4 d\left(\frac{1}{fX^I}\right) = -d\beta^J \wedge d\beta^K + 2g \cosh \varphi_I d\omega \wedge J + 4g^2 \sinh^2 \varphi_I f^{-2} X^I *_4 1. \tag{3.30}$$

This suggests that we introduce three independent functions

$$H_I = \frac{1}{fX^I}, \tag{3.31}$$

so that the second-order equation of motion becomes

$$d *_4 dH_I = -d\beta^J \wedge d\beta^K + 2g \cosh \varphi_I d\omega \wedge J + 4g^2 \sinh^2 \varphi_I H_J H_K *_4 1. \tag{3.32}$$

Note that the constraint $X^1 X^2 X^3 = 1$ indicates that the function f is given by

$$f = (H_1 H_2 H_3)^{-1/3}. \tag{3.33}$$

We have now found all of the constraints arising from supersymmetry and the equations of motion. To summarize, the solution is given by the metric

$$ds^2 = -(H_1 H_2 H_3)^{-2/3}(dt + \omega)^2 + (H_1 H_2 H_3)^{1/3} h_{mn} dx^m dx^n, \tag{3.34}$$

gauge potentials

$$A^I = -\frac{1}{H_I}(dt + \omega) + \beta^I, \tag{3.35}$$

vector multiplet scalars

$$X^I = \frac{(H_1 H_2 H_3)^{1/3}}{H_I}, \tag{3.36}$$

and hypermultiplet scalars φ_I. The metric h_{mn} on the base is Kähler, with anti-self-dual Kähler form J and holomorphic $(2,0)$-form Ω. The remaining quantities $(\varphi_I, \omega, \beta^I)$ must satisfy

$$
\begin{aligned}
(d\beta^I)^- &= gJ(H_J \cosh\varphi_K + H_K \cosh\varphi_J), \\
2d\omega^+ &= H_1(d\beta^1)^+ + H_2(d\beta^2)^+ + H_3(d\beta^3)^+, \\
\mathcal{R} &= -g\, d(\beta^1 \cosh\varphi_1 + \beta^2 \cosh\varphi_2 + \beta^3 \cosh\varphi_3), \\
d\varphi_I &= -2g \sinh\varphi_I J_m{}^n \beta_n^I dx^m,
\end{aligned}
\tag{3.37}
$$

as well as the equations of motion (3.32), which we repeat here:

$$
d *_4 dH_I = -d\beta^J \wedge d\beta^K + 2g \cosh\varphi_I d\omega \wedge J + 4g^2 \sinh^2\varphi_I H_J H_K *_4 1.
\tag{3.38}
$$

4. Supersymmetric solutions

From the above analysis, we see that the starting point for constructing supersymmetric solutions is the choice for the four-dimensional Kähler base. In this paper, we shall focus on the bi-axial case. However, for completeness, the first-order equations for the most general tri-axial ansatz for a cohomogeneity-one solution with S^3 orbits are presented in appendix B. In the bi-axial case, a gauge can be chosen such that the Kähler metric on the base is cast into the form

$$
ds_4^2 = \frac{dx^2}{4xh(x)} + \frac{x}{4}(\sigma_1^2 + \sigma_2^2 + h(x)\sigma_3^2),
\tag{4.1}
$$

where σ_i are SU(2) left-invariant 1-forms satisfying $d\sigma_1 = -\sigma_2 \wedge \sigma_3$. Corresponding to this metric, we introduce a natural vierbein basis

$$
e^1 = \frac{dx}{2\sqrt{xh}}, \qquad e^2 = \frac{\sqrt{x}}{2}\sigma_1, \qquad e^3 = \frac{\sqrt{x}}{2}\sigma_2, \qquad e^4 = \frac{\sqrt{xh}}{2}\sigma_3.
\tag{4.2}
$$

This base admits an anti-self-dual Kähler form

$$
J = \frac{1}{4}d(x\sigma_3) = e^1 \wedge e^4 - e^2 \wedge e^3,
\tag{4.3}
$$

and has the Ricci form

$$
\mathcal{R} = d\Big((2 - xh' - 2h)\sigma_3\Big) = 2\left(h' + \frac{2}{x}(h-1)\right)e^2 \wedge e^3 - 2(xh'' + 3h')\,e^1 \wedge e^4.
\tag{4.4}
$$

In addition to the Kähler metric on the base, we also make an ansatz for the 1-form ω, as well as the gauge functions β^I,

$$
\omega = w_1\sigma_1 + w_2\sigma_2 + w_3\sigma_3, \qquad \beta^I = U_1^I\sigma_1 + U_2^I\sigma_2 + U_3^I\sigma_3.
\tag{4.5}
$$

A true bi-axial solution, such as the black holes of [5, 6], will have only the components proportional to σ_3 turned on. However, by allowing non-trivial σ_1 and σ_2 components, we may also develop solutions asymptotic to deformed AdS$_5$, as investigated in [17, 18].

Note that while the base metric (4.1) preserves $\mathrm{SU}(2)_L \times \mathrm{U}(1)$ isometry, the complete five-dimensional solution only preserves a reduced $\mathrm{SU}(2)_L$ isometry unless all the σ_1 and σ_2 components vanish in (4.5).

We find that $d\omega$ decomposes into self-dual and anti-self-dual components according to

$$(d\omega)^{\pm} = 2\sqrt{h}\Big(w_1' \mp \frac{w_1}{xh}\Big)(e^1 \wedge e^2 \pm e^3 \wedge e^4) + 2\sqrt{h}\Big(w_2' \mp \frac{w_2}{xh}\Big)(e^1 \wedge e^3 \mp e^2 \wedge e^4)$$
$$+2\Big(w_3' \mp \frac{w_3}{x}\Big)(e^1 \wedge e^4 \pm e^2 \wedge e^3). \tag{4.6}$$

Similarly, $(d\beta^I)^{\pm}$ has the same form as $(d\omega)^{\pm}$, except with $w_i \to U_i^I$. In this case, the first-order supersymmetry equations (3.37) (or equivalently the first-order tri-axial equations (B.12)) reduce to

$$\varphi_I' = -\frac{2g}{xh}U_3^I \sinh\varphi_I,$$
$$(xU_3^I)' = \frac{gx}{2}(H_J \cosh\varphi_K + H_K \cosh\varphi_J),$$
$$U_j^{I'} = -\frac{U_j^I}{xh},$$
$$\Big(\frac{w_3}{x}\Big)' = \frac{1}{2}\sum_I H_I \Big(\frac{U_3^I}{x}\Big)',$$
$$w_j' - \frac{w_j}{xh} = \sum_I H_I U_j^{I'},$$
$$(x^2 h)' = 2x + 2gx\sum_I U_3^I \cosh\varphi_I, \tag{4.7}$$

as well as the algebraic conditions

$$\sum_I U_j^I \cosh\varphi_I = 0, \qquad gU_j^I \sinh\varphi_I = 0, \tag{4.8}$$

where $j = 1, 2$. The second-order equation of motion (3.38) can be expressed as

$$0 = \Big[x^2 h H_I' + 4\sum_{i=1}^{3} U_i^J U_i^K\Big]' - 2g\cosh\varphi_I (xw_3)' + g^2 \sinh^2\varphi_I x H_J H_K. \tag{4.9}$$

This may be rewritten as

$$0 = \Big[x^2 h H_I' + 4\sum_{i=1}^{3} U_i^J U_i^K - 2g\cosh\varphi_I xw_3\Big]' + g^2 \sinh^2\varphi_I \Big(x H_J H_K - 4\frac{w_3}{h}U_3^I\Big), \tag{4.10}$$

where we have used the first-order equation for φ_I.

Note that, just as in [5, 6], we could have chosen the opposite sign for the Kähler form in (4.3). This simply corresponds to taking

$$w_i \to -w_i, \qquad U_i^I \to -U_i^I, \tag{4.11}$$

in the expressions above.

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5. Solutions without hyperscalars

We are principally interested in obtaining and classifying all solutions of the supersymmetric bi-axial system given by the first-order equations (4.7), algebraic constraints (4.8) and equation of motion (4.10). To proceed, we first consider the case when the hypermatter scalars φ_I are set to zero. This case corresponds to the gauged supergravity version of the STU model, and has been extensively studied. Nevertheless, as shown below, there are still surprises to be found when analyzing these solutions.

By setting $\varphi_I = 0$, the above system of equations reduces to

$$(xU_3^I)' = \frac{gx}{2}(H_J + H_K),$$
$$\left(\frac{w_3}{x}\right)' = \frac{1}{2}\sum_I H_I \left(\frac{U_3^I}{x}\right)',$$
$$(x^2 h)' = 2x + 2gx\sum_I U_3^I, \tag{5.1}$$

involving the σ_3 components, and

$$U_j^{I\,\prime} = -\frac{U_j^I}{xf},$$
$$w_j' - \frac{w_j}{xh} = \sum_I H_I U_j^{I\,\prime},$$
$$0 = \sum_I U_j^I \tag{5.2}$$

$(j = 1, 2)$ involving the σ_1 and σ_2 components. In addition, the second-order equation reduces to

$$0 = \left[x^2 h H_I' + 4\sum_{i=1}^{3} U_i^J U_i^K - 2gxw_3\right]', \tag{5.3}$$

which admits a first integral that is proportional to the Noether electric charge Q^I of the gauge fields. Note that this equation of motion is the only expression coupling the σ_1 and σ_2 components U_j^I to the functions H_I.

We may generate a formal solution to the above system by assuming the functions H_I to be arbitrary. The functions U_3^I, w_3 and h can then be obtained by successive integration of the first-order equations in (5.1). Similarly, the functions U_j^I and w_j follow from (5.2) by integration. At this stage, all quantities may now be formally written in terms of H_I and its integrals. Inserting these expressions into the (5.3) then gives rise to a set of integro-differential equations whose solutions correspond to generically 1/4 BPS configurations solving all equations of motion. However, in practice, such a formal solution is difficult to analyze. Hence, we instead turn to some explicit solutions.

5.1 Solutions with $\mathbb{R} \times \mathrm{SU}(2)_L \times \mathrm{U}(1)$ isometry

We recall that the bi-axial ansatz (4.1) involves a Kähler base with $\mathrm{SU}(2)_L \times \mathrm{U}(1)$ isometry. This isometry may be extended to the complete solution by taking $U_1^I = 0 = U_2^I$ and

$w_1 = 0 = w_2$, in which case the equations (5.2) are trivially satisfied. Together with time translational invariance, the full isometry of the solution is $\mathbb{R} \times \text{SU}(2)_L \times \text{U}(1)$.

Even in this case, however, an analytic form for the general solution is not apparent. Nevertheless, by assuming 'harmonic functions' of the form $H_I = 1 + q_I/x$, we find a class of solutions given by

$$
\begin{aligned}
H_I &= 1 + \frac{q_I}{x}, \\
U_3^I &= \frac{1}{2}g\left(x + q_J + q_K\right) + \frac{\alpha_I}{x}, \\
h &= 1 + g^2(x + \sum_I q_I) + \frac{2g\sum_I \alpha_I}{x} + \frac{\gamma}{x^2}, \\
w_3 &= \frac{1}{2}g\left(x + \sum_I q_I\right) + \frac{2\sum_I \alpha_I + g\sum_{I<J} q_I q_J}{4x} + \frac{\sum_I q_I\,\alpha_I}{3x^2}.
\end{aligned}
\tag{5.4}
$$

This solution is parameterized by the quantities q_I and α_I, $I = 1, 2, 3$ satisfying the condition

$$
q_1\,\alpha_1 = q_2\,\alpha_2 = q_3\,\alpha_3.
\tag{5.5}
$$

In this case, the constant γ may be expressed as

$$
\gamma = \frac{4\alpha_I \alpha_J}{q_K},
\tag{5.6}
$$

for any choice of $I \neq J \neq K$, so long as q_K is non-vanishing. (If all three charges q_I vanish, then γ is arbitrary.)

Alternatively, this solution can be reexpressed in terms of the H_I functions as

$$
\begin{aligned}
H_I &= 1 + \frac{q_I}{x}, \\
U_3^I &= \frac{g}{2}x H_J H_K + \frac{\gamma_I}{x}, \\
h &= 1 + g^2 x \prod_I H_I + \frac{2g\sum_I \gamma_I}{x} + \frac{4(g\gamma_1 q_1 + \gamma_2\gamma_3/q_1)}{x^2}, \\
w_3 &= \frac{g}{2}x\prod_I H_I + \frac{\sum_I \gamma_I}{2x} + \frac{\gamma_1 q_1}{x^2},
\end{aligned}
\tag{5.7}
$$

where

$$
\gamma_I \equiv \alpha_I - \frac{1}{2}g q_J q_K.
\tag{5.8}
$$

Note that the integration constants satisfy

$$
q_1\,\gamma_1 = q_2\,\gamma_2 = q_3\,\gamma_3.
\tag{5.9}
$$

As a result, the last terms in the expressions for h and w_3 are in fact symmetric in the charges. As we shall see, both of the above sets of expressions will be useful for exploring various limits as well as generalizations of the solutions.

These solutions generically preserve 1/4 of the supersymmetry of the $D = 5$, $\mathcal{N} = 2$ gauged supergravity. The mass, angular momentum and R-charges are given by

$$M = 2gJ + \frac{1}{4}(q_1 + q_2 + q_3) - \frac{1}{4}g(\alpha_1 + \alpha_2 + \alpha_3) + \frac{1}{8}g^2(q_1q_2 + q_1q_3 + q_2q_3),$$

$$J = -\frac{1}{4}(\alpha_1 + \alpha_2 + \alpha_3) + \frac{1}{8}g(2\gamma + q_1q_2 + q_1q_3 + q_2q_3) - \frac{1}{3}g^2(\alpha_1q_1 + \alpha_2q_2 + \alpha_3q_3)$$
$$+ \frac{1}{4}g^3q_1q_2q_3,$$

$$Q_I = \frac{1}{4}q_I - \frac{1}{4}g(\alpha_J + \alpha_K - \alpha_I) + \frac{1}{8}g^2(q_I(q_J + q_K) - q_Jq_K), \tag{5.10}$$

or equivalently

$$M = 2gJ + \frac{1}{4}(q_1 + q_2 + q_3) - \frac{1}{4}g(\gamma_1 + \gamma_2 + \gamma_3),$$

$$J = -\frac{1}{4}(\gamma_1 + \gamma_2 + \gamma_3) + g\gamma_2\gamma_3/q_1,$$

$$Q_I = \frac{1}{4}q_I - \frac{1}{4}g(\gamma_J + \gamma_K - \gamma_I). \tag{5.11}$$

Note that in presenting the mass, charge and angular momentum results, we suppress a common factor that is the volume of spatial principal orbits, which can be S^3, or a lens space $S^3/\mathbb{Z}_k$, for some integer k, which is fixed by a specific regularity requirement of the solutions. It is easy to see that these quantities satisfy the BPS condition

$$M = 2gJ + Q_1 + Q_2 + Q_3. \tag{5.12}$$

It should be noted that black holes in five dimensions may carry two independent angular momenta, J_1 and J_2. Our choice of a cohomogeneity-one base, however, restricts the system to two equal angular momenta, $J_1 = J_2 = J$. In general, the solution becomes non-rotating when $\alpha_I = \frac{1}{2}gq_Jq_K$, or equivalently when $\gamma_I = 0$.

The non-rotating solutions with $\gamma_I = 0$ are in fact the original superstars of [2, 3]. These have naked singularities at $x = 0$. On the other hand, the supersymmetric black holes of Gutowski and Reall [5, 6] are recovered when $\alpha_I = 0$. In this case, the radial coordinate x runs from the horizon at $x = 0$, where the geometry is a direct product of AdS$_2$ and a squashed S^3, to asymptotic AdS$_5$ as $x \to \infty$. The three-equal-charge case of the solution (5.4) was found in [17], while the general case was obtained in [28].

In general, the solution (5.4) describes a spacetime in which there is a region with closed timelike curves (CTC's). Such a spacetime is sometimes referred to as a 'time machine.' In this case, x runs from $x_0 > 0$, where x_0 is the greatest root of f, to asymptotic infinity. These time-machine solutions can be made perfectly regular with appropriate assignments of the periodicity for the real time coordinate t, as discussed in [28]. Naked CTC's can be avoided by imposing the additional condition that $w_3(x_0) = 0$. This leads to the supersymmetric solitons that are discussed below.

5.1.1 Massless solitons

The properties of the solitons are largely determined by the parameters q_I. We shall first consider the case of the solution given by (5.4) with only a single U(1) gauge field active.

This corresponds to having $q_2 = 0 = q_3$, $\alpha_1 = 0$ and $\alpha_2 = \alpha_3 \equiv c_1$. Let us choose the parameters q_1 and c_1 so that

$$h(x_0) = 0, \qquad w_3(x_0) = 0. \tag{5.13}$$

The first condition is needed in order to avoid power-law curvature singularities, while the second one ensures that there are no CTC's, as we have discussed earlier. These conditions can be satisfied by setting

$$q_1 = -\frac{g^2 x_0^2}{1 + g^2 x_0}, \qquad c_1 = -\frac{g\, x_0^2}{2(1 + g^2 x_0)}. \tag{5.14}$$

This implies that $q_1 = 2g\, c_1$. Now we have

$$
\begin{aligned}
H_1 &= \frac{x + g^2 x_0 (x - x_0)}{x + g^2 x\, x_0}, \\
w_3 &= \frac{g(x - x_0)(x + x_0 + g^2 x\, x_0)}{2x(1 + g^2 x_0)}, \\
h &= \frac{(x - x_0)(x + x_0 + g^2 x\, x_0)(1 + g^2 x)}{x^2(1 + g^2 x_0)},
\end{aligned}
\tag{5.15}
$$

and indeed $x_0 > 0$ is the greatest root of f. It follows that the solution does not have a power-law curvature singularity for $x \geq x_0$ with $x_0 > 0$. In addition, there are no CTC's since we have

$$g_{\psi\psi} = \frac{(x - x_0)(x + x_0 + g^2 x\, x_0)}{4x(1 + g^2 x_0) H_1^{2/3}} \geq 0. \tag{5.16}$$

The consequence of this is that t is a globally defined time coordinate, in that for any constant t, the spacetime is foliated by spatial sections.

In order for the (x, ψ) subspace to form a smooth $\mathbb{R}^2$ at $x = x_0$, the period of the angular coordinate ψ must be

$$\Delta\psi = \frac{4\pi}{2 + g^2 x_0}. \tag{5.17}$$

In addition, in order for the level surfaces of the principal orbits to be regular, the period of ψ must be such that

$$\Delta\psi = \frac{4\pi}{k}, \tag{5.18}$$

for some integer k. As a consequence, the principal orbits are lens spaces $S^3/\mathbb{Z}_k$. Therefore, in order to avoid a conical singularity, x_0 is fixed to be

$$x_0 = \frac{k - 2}{g^2}, \tag{5.19}$$

for each lens space $S^3/\mathbb{Z}_k$. The requirement of $x_0 > 0$ implies that we must have $k \geq 3$.

It is easy to verify using (5.10) that the mass, charge and angular momentum all vanish for this soliton, when the conditions (5.14) for the regularity and the absence of CTC's are imposed. In this sense, it provides an explicit example of a 'texture' in gauged supergravity. Let us be more precise about this, since from the gravitational point of view one can always add an arbitrary constant to the mass. Throughout this paper, we shall take the mass M_{AdS} of the AdS vacuum to be zero, since the CFT Casimir energy is not relevant for our discussion. Then, by zero mass we mean specifically that $M = M_{\mathrm{AdS}}$.

5.1.2 Massive solitons

We now consider the case for which $q_3 = 0$ and q_1 and q_2 are nonvanishing. Then we must have $\alpha_1 = \alpha_2 = 0$ by virtue of (5.5). Let us choose the parameters q_1, q_2 and α_3 such that the conditions given by (5.13) are satisfied. This can be achieved by setting

$$g^2 = \frac{x_0}{(x_0 + q_1)(x_0 + q_2)}, \qquad \alpha_3 = -\frac{1}{2}g(2x_0^2 + 2x_0(q_1 + q_2) + q_1 q_2). \tag{5.20}$$

It follows that we have

$$H_1 = 1 + \frac{q_1}{x}, \qquad H_2 = 1 + \frac{q_2}{x}, \qquad H_3 = 1,$$
$$h = \frac{(x - x_0)[xx_0 + 2x_0^2 + 2x_0(q_1 + q_2) + q_1 q_2]}{x(x_0 + q_1)(x_0 + q_2)},$$
$$w_3 = \frac{(x - x_0)(x + x_0 + q_1 + q_2)}{2x}. \tag{5.21}$$

We can verify that the solution does not have a power-law curvature singularity for $x \geq x_0$, where $x_0 > \max\{0, -q_1, -q_2\}$. There are no CTC's either, since we have

$$g_{\psi\psi} = \frac{(x - x_0)[x^2 + x(x_0 + q_1 + q_2) + (x_0 + q_1)(x_0 + q_2)]}{4x^2(H_1 H_2)^{2/3}} \geq 0. \tag{5.22}$$

The (x,ψ) subspace forms an $\mathbb{R}^2$ near $x = x_0$ if the period of ψ is

$$\Delta\psi = \frac{4\pi(x_0 + q_1)(x_0 + q_2)}{3x_0^2 + 2x_0(q_1 + q_2) + q_1 q_2}. \tag{5.23}$$

In order for the level surfaces of the principal orbits to be regular, i.e., $S^3/\mathbb{Z}_k$, the period of the angle ψ has to be $\Delta\psi = \frac{4\pi}{k}$. Thus, we have

$$k = \frac{3x_0^2 + 2x_0(q_1 + q_2) + q_1 q_2}{(x_0 + q_1)(x_0 + q_2)}. \tag{5.24}$$

Note that there is no solution for $k = \pm 1$ that satisfies the regularity conditions. For $k = 2$, we have $x_0 = \sqrt{q_1 q_2}$ which, together with (5.20), implies that q_1 and q_2 must both be positive. The other values of k can only be achieved with at least one of the q_i's negative.

The mass, charge and angular momentum for this solitonic solution are given by

$$J = \frac{x_0}{4g}, \qquad\qquad M = \frac{1}{4}(3x_0 + q_1 + q_2).$$
$$Q_1 = \frac{1}{4}(q_1 - x_0), \qquad Q_2 = \frac{1}{4}(q_2 - x_0), \qquad Q_3 = \frac{x_0}{4}, \tag{5.25}$$

which of course satisfy the BPS condition (5.12). This charged rotating soliton has a positive mass.

5.1.3 Negative mass solitons

Finally, we consider the case in which none of the q_i vanish. We can take $\alpha_I = g\beta/q_I$, for a constant β. In order for the conditions given by (5.13) to be satisfied, we take

$$g = \frac{x_0 \prod_{I<J} q_I q_J + 2q_1 q_2 q_3}{\sqrt{\prod_I (x_0 + q_I)(\prod_{I<J} q_I^2 q_J^2 - 2q_1 q_2 q_3(2x_0 + \sum_I q_I))}},$$

$$\beta = -\frac{q_1 q_2 q_3 x_0(2x_0^2 + 2x_0 \sum_I q_I + \prod_{I<J} q_I q_J)}{2(x_0 \prod_{I<J} q_I q_J + 2q_1 q_2 q_3)}. \tag{5.26}$$

The local expressions for this class of solutions were obtained in [28] by taking the BPS limit of the non-extremal rotating black hole solutions constructed in [29, 30]. Here, we analyse the solutions in more detail, and demonstrate that smooth solutions with negative mass can also arise. Since the resulting expressions for the metric functions are rather long, we shall examine only a couple of particular cases.

Single charge. We first consider $q_2 = q_3 = 2q_1 \equiv -2q$, in which case the expressions become significant simpler. The soliton condition (5.13) implies that

$$x_0 = 2q + \frac{2\sqrt{q}}{g}, \qquad \beta = \frac{2q^2}{g^2}(g^2 q - 1), \tag{5.27}$$

which requires that $q > 0$. Consequently, we have

$$H_1 = 1 - \frac{q}{x}, \qquad H_2 = H_3 = 1 - \frac{2q}{x},$$

$$h = \frac{(x + x_0 - 4q)(4q(x - x_0) + x_0^2)(x - x_0)}{(x_0 - 2q)^2 x^2},$$

$$w = -\frac{\sqrt{q}(x - q)(x + x_0 - 4q)(x - x_0)}{(x_0 - 2q)x^2}. \tag{5.28}$$

Thus, the solutions do not have a power-law curvature singularity for $x \geq x_0$ and $0 < q < x_0/2$. There are also no CTC's, since we have

$$g_{\psi\psi} = \frac{(x - q)(x + x_0 - 4q)(x - x_0)}{4x^2(H_1 H_2 H_3)^{2/3}} \geq 0. \tag{5.29}$$

The (x,ψ) subspace forms an $\mathbb{R}^2$ near $x = x_0$ if the period of ψ is

$$\Delta\psi = \frac{2\pi(x_0 - 2q)}{x_0}. \tag{5.30}$$

Combining this with the usual requirement that $\Delta\psi = 4\pi/k$ and with the condition (5.27) yields

$$x_0 = \frac{k(k - 2)}{2g^2}, \qquad q = \frac{(k - 2)^2}{4g^2}. \tag{5.31}$$

Thus, we must have $k \geq 3$.

For these solitonic solutions, $M = Q_1 = -q/4$ and $Q_2 = Q_3 = J = 0$. Thus, we see that although all three U(1) gauge fields $A_{(1)}^I$ are turned on, there is only one charge.

The solution has zero angular momentum although it has rotations. It is furthermore rather surprising that regularity and the absence of CTC's implies that these solitons have negative mass, or more specifically $M < M_{\text{AdS}}$. Solutions with negative mass have been referred to as 'phantom matter,' whose repulsive behavior may be useful for modeling the observed acceleration of the scale factor $a(t)$ of the universe [31, 32].

Positive mass theorems in general relativity have established that asymptotically AdS solutions of the Einstein equations with physically acceptable matter sources cannot have negative total mass. For instance, the negative mass Schwarzschild solution has a naked power-law curvature singularity. Our solutions evade such positive mass theorems by having an asymptotic geometry of $\text{AdS}_5/\mathbb{Z}_k$ with $k \geq 3$, rather than AdS_5. To be more precise, the S^3 within AdS_5 has been replaced by the lens space $S^3/\mathbb{Z}_k$. Since these solutions are supersymmetric, they are perturbatively stable against local energy fluctuations.

Three equal charges. Another simple example is the case of three equal charges, for which we can set $q_I \equiv q$. While this case has already been discussed in detail in [28], the possibility of negative mass solitons was not realized. Again, after imposing the condition (5.13) we have

$$g = \frac{\sqrt{-q}(2q + 3x_0)}{(q + x_0)\sqrt{(q + x_0)(3q + 4x_0)}}, \qquad \beta = \frac{qx_0(3q^2 + 6qx_0 + 2x_0^2)}{4q + 6x_0}, \qquad (5.32)$$

and

$$H_1 = H_2 = H_3 = 1 + \frac{q}{x},$$
$$h = \frac{[-q(3x_0 + 2q)^2 x^2 + (3q^2 + 6qx_0 + 2x_0^2)(2x_0^2 - 3qx_0 - 3q^2)x + (3q^2 + 6qx_0 + 2x_0^2)^2 x_0](x - x_0)}{(4x_0 + 3q)(x_0 + q)^3 x^2},$$
$$w = \frac{\sqrt{-q}[q(2x^2 + 11x_0 x + 2x_0^2 + 3q^2) + 3(2q^2 + x_0 x)(x + x_0)](x - x_0)}{2(q + x_0)\sqrt{(q + x_0)(3q + 4x_0)} \, x^2}. \qquad (5.33)$$

There are two cases for which the solution is real and completely regular for $x \geq x_0$:

$$\begin{aligned}
\text{Case I}: \quad & x_0 < 0, \qquad q > 0, \qquad -q < x_0 < -\frac{3}{4}q, \\
\text{Case II}: \quad & x_0 > 0, \qquad q < 0, \qquad -q < x_0.
\end{aligned} \qquad (5.34)$$

Note that the possibility that x_0 can be negative for regular solutions can only arise when all three charge parameters q_I are non-vanishing, since otherwise negative x would lead to a power-law curvature singularity at $x = 0$. This can be seen easily by noting that the radius square of the S^2 of the base space is given by $x(H_1 H_2 H_3)^{1/3}$, which is a constant at $x = 0$ for the case with q_I all non-vanishing, whilst becomes zero for the cases when at least one of the q_I vanishes.

For both cases (5.34), we have verified that there are no CTC's since

$$g_{\psi\psi} = \frac{[(4x_0 + 3q)x^2 + (4x_0 + 3q)(x_0 + 3q)x + (9qx_0 + 5q^2 + 3x_0^2)q](x - x_0)}{4(4x_0 + 3q)(x + q)^2} \geq 0. \quad (5.35)$$

The (x,ψ) subspace forms an $\mathbb{R}^2$ near $x = x_0$ if the period of ψ is

$$\Delta\psi = \frac{4\pi(x_0 + q)(4x_0 + 3q)}{(8x_0 + 5q)x_0} \,. \tag{5.36}$$

Combining this with the requirement that $\Delta\psi = 4\pi/k$ yields

$$x_0 = \begin{cases} x_0^\pm = \frac{5-7k\pm\sqrt{(k+1)(k+25)}}{8(k-2)}q, & \text{for } k \neq 2; \\ -\frac{2}{3}q, & \text{for } k = 2. \end{cases} \tag{5.37}$$

In order to satisfy the conditions for Case I, we find that $k \leq -25$. Interestingly enough, these conditions are met for either sign in x_0. The boundaries $x_0 = -q$ and $x_0 = -\frac{3}{4}q$ are saturated for $k \to -\infty$ by $x_0 = x_0^-$ and $x_0 = x_0^+$, respectively. On the other hand, the conditions for Case II are satisfied only for $x_0 = x_0^-$ with $k \geq 3$. Then the boundary $x_0 = -q$ is saturated for $k \to +\infty$.

For these solitonic solutions,

$$M = \frac{(12x_0^2 + 5q^2 + 15qx_0)q}{4(4x_0 + 3q)^2} \,, \qquad J = -\frac{1}{4}\left(-\frac{(x_0 + q)q}{4x_0 + 3q}\right)^{3/2} , \qquad Q_i = \frac{(x_0 + q)q}{4(4x_0 + 3q)} \,. \tag{5.38}$$

These three-equal charge solitons have positive mass for Case I and negative mass for Case II. As in the previous case, the negative-mass solitons evade the positive mass theorems by being asymptotically $\mathrm{AdS}_5/\mathbb{Z}_k$, where $k \geq 3$.

5.2 Solutions with $\mathbb{R} \times \mathrm{SU}(2)_L$ isometry

Returning to the first-order equations (5.1) and (5.2), we see that the above system always admits an $\mathbb{R} \times \mathrm{SU}(2)_L \times \mathrm{U}(1)$ breaking deformation where w_1 and w_2 (multiplying σ_1 and σ_2 in the time fibration, respectively) are turned on. By keeping $U_1^I = 0 = U_2^I$, this deformation is essentially restricted to the metric. In particular, the equation of motion (5.3) is left unchanged. This deformation reduces the $\mathbb{R} \times \mathrm{SU}(2)_L \times \mathrm{U}(1)$ isometry of the five-dimensional metric to $\mathbb{R} \times \mathrm{SU}(2)_L$ only, although the Kähler base is undeformed and retains the full original isometry.

Integrating the second equation in (5.2), we see that the solution given by (5.4) (or equivalently (5.7)) can be further generalized to include σ_1 and σ_2 in the timelike fibration as follows:

$$w_1 = c_1 u \,, \qquad w_2 = c_2 u \,, \qquad u = u_0 \exp\left[\int_{x_0}^x \frac{dx'}{x'f(x')}\right]. \tag{5.39}$$

These solutions still preserve $1/4$ of the supersymmetry of the $D = 5$, $\mathcal{N} = 2$ gauged supergravity. For the w_i generalisation of the AdS rotating black holes, corresponding to q_I all equal and $\alpha_I = 0$, this reduces to a family of solutions constructed in [17]. Furthermore, for $c_2 = 0$ and $q_I = \alpha_I = 0$, these solutions reduce to the deformations of AdS_5 constructed in [15] and further analyzed in [18]. We can extend those solutions to include c_2, for which the absence of CTC's in the Gödel-like universe at asymptotic infinity can be achieved by requiring

$$c_1^2 + c_2^2 \leq \frac{1}{4g^2} \,. \tag{5.40}$$

In obtaining this result, we have normalized u by choosing an appropriate u_0 such that $u = 1$ for $x = \infty$. We shall use the same normalization for u for other solutions as well. The criteria for avoiding CTC's at $x = x_0$ are the same as for the solutions in the previous subsection.

For general q_I, u can be expressed in terms of a sum of polynomial roots. Note that, for solutions where x runs from $x = 0$ to ∞, the function u runs from 0 at $x = 0$ to 1 at $x = \infty$. It is clear that there are no CTC's near $x = 0$, while CTC's at large x can also be avoided by taking the condition (5.40). For example, for the w_i generalisation of rotating black holes, we have

$$u = \left(1 + \frac{q_1 + q_2 + q_3 + g^{-2}}{x}\right)^{-\frac{1}{1+g^2(q_1+q_2+q_3)}} . \tag{5.41}$$

It is easy to verify that there are no CTC's provided that (5.40) is satisfied. For solutions where x runs from $x = x_0 > 0$ to ∞, the function u runs from 0 at $x = x_0$ to 1 at $x = \infty$. Thus, again there are no CTC's near $x = x_0$ and the condition for the absence of CTC's at infinity is the same as (5.40). The general expression for u can be complicated. In the special case of vanishing q_3, we find a simple expression, given by

$$u = \left(\frac{A - B - 2g^2 x}{A + B + 2g^2 x}\right)^{1/A} , \tag{5.42}$$

where

$$A \equiv \sqrt{1 + 2g^2(q_1 + q_2) + g^4(q_1 - q_2)^2 - 8g^3 \gamma_3} , \qquad B \equiv 1 + g^2(q_1 + q_2) . \tag{5.43}$$

Finally, we present the explicit expression for u for the massless soliton studied in section 5.1.1. It is given by

$$u = \frac{g^2(x - x_0)^{\frac{1}{2+g^2 x_0}} \left(x + \frac{x_0}{1+g^2 x_0}\right)^{\frac{1+g^2 x_0}{2+g^2 x_0}}}{1 + g^2 x} . \tag{5.44}$$

6. Solutions with hyperscalars

In this section, we study supersymmetric solutions with the hypermatter scalars φ_I turned on. We obtain some new explicit analytical solutions as well as a class of new numerical solutions.

6.1 Solutions with $\mathbb{R} \times \mathrm{SU}(2)_L$ isometry

In [10], a general class of static bubble solutions were obtained for the STU model coupled to the three hypermatter scalars φ_I. Even with these additional scalars turned on, the first-order equations (4.7) still allow w_1 and w_2 to be turned on without affecting the equation of motion (4.10). As a result, this class of bubble solutions admits a $\mathrm{SU}(2)_L \times \mathrm{U}(1)$ breaking

deformation of the form

$$\cosh \varphi_I = (xH_I)',$$
$$h = 1 + g^2 xH_1H_2H_3,$$
$$U_3^I = \frac{g}{2}xH_JH_K, \qquad U_1^I = U_2^I = 0,$$
$$w_3 = \frac{g}{2}xH_1H_2H_3, \qquad w_1 = c_1u, \qquad w_2 = c_2u, \tag{6.1}$$

where the functions H_I satisfy the equations

$$h(xH_I)'' = -g^2[(xH_I)'^2 - 1](H_1H_2H_3)H_I^{-1}, \tag{6.2}$$

and where the metric deformation function is given by

$$u = u_0 \exp\left[\int_{x_0}^x \frac{dx'}{x'h(x')}\right]. \tag{6.3}$$

For $c_1 = c_2 = 0$, this reduces to the AdS bubbles constructed in [10], which generalize a subset of 1/2 BPS LLM solutions [9] to 1/4 and 1/8 BPS solutions by turning on two and three independent U(1) fields, respectively. By relaxing the $SU(2)_L \times U(1)$ isometry of the AdS bubbles [10] to $SU(2)_L$ only, we find that there is a more general family of solutions, for which the c_1 and c_2 deformation parameters are non-zero.

In the single-charge case, with $H_2 = H_3 = 1$, there is an explicit expression for H_1:

$$H_1 = \sqrt{1 + \frac{2(1 + g^2q_1)}{g^2x} + \frac{c^2}{g^4x^2}} - \frac{1}{g^2x}. \tag{6.4}$$

Regularity of the AdS bubble requires that $c = 1$. In this case, the deformation function u becomes

$$u = \frac{g^2(2 + g^2q_1)x}{1 + g^2(1 + g^2q_1)x + \sqrt{(1 + g^2x)^2 + 2g^4q_1x}}. \tag{6.5}$$

The geometry runs from a timelike bundle over $\mathbb{R}^4$ at short distance ($x = 0$) to a Gödel-like universe asymptotically ($x \to \infty$). Since u and w_3 vanish linearly at $x = 0$, it follows that the solution does not have CTC's near $x = 0$. When $x \to \infty$, the absence of CTC's requires the same condition as in (5.40). It is straightforward to verify that there are no CTC's from $x = 0$ to ∞ when the above condition is satisfied.

For the generic three-charge situation, the equations (6.2) do not seem to allow solutions to be found explicitly. (The numerical analysis was performed in [38].) However, it is easy to see that the structure of the three-charge solution is rather similar to that of the single-charge case. The coordinate runs from $x = 0$ to ∞, with H_I and f being certain constants at $x = 0$. It follows that the w_i vanish at $x = 0$, implying no CTC's near $x = 0$. Since $H_I \sim 1 + q_I/x$ for large x, for sufficiently small $c_1^2 + c_2^2$, CTC's can be avoided in the asymptotic region. Then such bubbling solitonic solutions are completely regular and are free of CTC's.

6.2 General rotating bubbles with $\mathbb{R} \times \mathrm{SU}(2)_L \times \mathrm{U}(1)$ isometry

In the previous subsection, we obtained analytical solutions by imposing the condition $\cosh \varphi_I = (xH_I)'$, which was originally given in [10]. Although this is a necessary condition for static bubbles, it is not a direct consequence of supersymmetry and hence can be relaxed when the system is rotating. Here we consider the general system with non-vanishing hypermatter scalars φ_I. However, we restrict our attention to solutions with $\mathbb{R} \times \mathrm{SU}(2)_L \times \mathrm{U}(1)$ isometry such that the metric can be expressed in a non-rotating frame in the asymptotic region. This corresponds to setting $U_1^I = U_2^I = 0$ and $w_1 = w_2 = 0$, but leaving all other fields free up to the first-order equations (4.7)

$$
\varphi_I' = -\frac{2g}{xh} U_3^I \sinh \varphi_I,
$$
$$
(xU_3^I)' = \frac{gx}{2}(H_J \cosh \varphi_K + H_K \cosh \varphi_J),
$$
$$
\left(\frac{w_3}{x}\right)' = \frac{1}{2}\sum_I H_I \left(\frac{U_3^I}{x}\right)',
$$
$$
(x^2 h)' = 2x + 2gx \sum_I U_3^I \cosh \varphi_I, \tag{6.6}
$$

and second-order equations (4.10)

$$
0 = \left[x^2 h H_I' + 4U_3^J U_3^K - 2g \cosh \varphi_I x w_3 \right]' + g^2 \sinh^2 \varphi_I \left(x H_J H_K - 4\frac{w_3}{h} U_3^I \right). \tag{6.7}
$$

In general, there are two types of solitonic solutions. The first type can be referred to as $\mathbb{R}^4$ solitons, where the coordinate x runs from 0 to ∞. This is because the geometry near $x = 0$ is a direct product of time and $\mathbb{R}^4$. It can be demonstrated numerically that such solutions exist. In order to do so, we may first show that the above system admits a regular Taylor series solution near $x = 0$ of the form

$$
H_I = h_I^0 - \frac{g^2 x}{4}\Big[(h_I^0)^2 \cosh \varphi_J^0 \cosh \varphi_K^0 + h_J^0 h_K^0 (1 + 3\sinh^2 \varphi_I^0)
$$
$$
+ \cosh \varphi_I^0 (h_I^0(h_J^0 \cosh \varphi_K^0 + h_K^0 \cosh \varphi_J^0) - 8\gamma)\Big] + \cdots,
$$
$$
\cosh \varphi_I = \cosh \varphi_I^0 - \frac{g^2 x}{2} \sinh^2 \varphi_I^0 (h_J^0 \cosh \varphi_K^0 + h_K^0 \cosh \varphi_J^0) + \cdots,
$$
$$
U_3^I = \frac{gx}{4}\Big[h_J^0 \cosh \varphi_K^0 + h_K^0 \cosh \varphi_J^0 \Big] - \frac{g^2 x^2}{24}\Big[-16\gamma \cosh \varphi_J^0 \cosh \varphi_K^0
$$
$$
+ 2h_I^0(h_J^0 \cosh \varphi_J^0(1 + 3\sinh^2 \varphi_K^0) + h_K^0 \cosh \varphi_K^0 (1 + 3\sinh^2 \varphi_J^0))
$$
$$
+ \cosh \varphi_I^0 \big((h_J^0)^2(1 + 3\sinh^2 \varphi_K^0) + (h_K^0)^2(1 + 3\sinh^2 \varphi_J^0)
$$
$$
+ 2h_J^0 h_K^0 \cosh \varphi_J^0 \cosh \varphi_K^0 \big)\Big] + \cdots, \tag{6.8}
$$

along with

$$h = 1 + \frac{g^2 x}{3} \sum_I h_I^0 \cosh \varphi_J^0 \cosh \varphi_K^0 + \cdots ,$$

$$w_3 = g\gamma x - \frac{g^3 x^2}{48} \left[-16\gamma \sum_I h_I^0 \cosh \varphi_J^0 \cosh \varphi_K^0 + 3 \prod_I (h_J^0 \cosh \varphi_K^0 + h_K^0 \cosh \varphi_J^0) \right.$$

$$\left. + 6 \sum_I h_I^0 \cosh \varphi_I^0 ((h_J^0)^2 \sinh^2 \varphi_K^0 + (h_K^0)^2 \sinh^2 \varphi_J^0) \right] + \cdots . \quad (6.9)$$

In general, the solution to the system (6.6) and (6.7) may be specified by 14 independent parameters. However, regularity at the origin reduces this to the 7 parameters $(\varphi_I^0, h_I^0, \gamma)$. Numerical integration may then be used to connect this solution to its most general counterpart developed around $x = \infty$.

Since we find that the asymptotic solution at $x = \infty$ is well behaved, we see that regular bubbling solutions may be obtained for generic values of the 7 parameters $(\varphi_I^0, h_I^0, \gamma)$. However, it should be noted that logarithmic terms are almost always present in the expansion. The presence of such terms gives rise to potentially infinite mass for these rotating solitons. Since mass can be extracted from the behavior of the metric at the asymptotic boundary, this infinite mass is closely related to deformations of the S^3 at infinity which in turn leads to a deformation of the global spacetime away from asymptotic AdS$_5$.

Furthermore, while smooth bubbling solutions exist for a large range of parameters, they generally contain CTC's. However, by adjusting the initial parameters $(\varphi_I^0, h_I^0, \gamma)$ appropriately, we find that solutions without CTC's may be obtained.

We now present the analysis for the single charge $\mathbb{R}^4$ soliton in somewhat more detail. To obtain a single charge solution, we set $H_2 = H_3 = 1$ as well as $\varphi_2 = \varphi_3 = 0$. To avoid generating magnetic field components for A^2 and A^3, we must also set $U_3^2 = U_3^3 = w_3$. The remaining non-trivial fields may then be given in terms of two functions $H(x)$ and $\zeta(x)$:

$$H_1 = H,$$
$$\cosh \varphi_1 = (xH)' + \frac{1}{x}(x^2\zeta)'',$$
$$U_3^1 = \frac{gx}{2},$$
$$w_3 = \frac{gx}{2}H + \frac{g}{2x}(x^2\zeta)',$$
$$h = 1 + g^2 xH + \frac{g^2}{x^2}(x^3\zeta)'. \quad (6.10)$$

In this single charge case, the combination of the first and second order equations above reduce to a coupled set of two equations which contain up to second derivatives of H and third derivatives of ζ. This indicates that the general solution may be specified by five parameters. An expansion at infinity gives

$$H = 1 + \frac{1}{g^2 x}(h_1 + h_{11} \log x) + \frac{1}{g^4 x^2} h_2 + \cdots , \quad (6.11)$$

$$g^2 \zeta = -\frac{1}{2} h_{11} + \frac{1}{g^2 x}\left(f_1 + h_{11}\left(1 - \frac{1}{2}h_{11}\right) \log x \right) + \frac{1}{g^4 x^2}\left(f_2 + h_{11}\left(1 - \frac{1}{2}h_{11}\right)^2 \log x \right) + \cdots ,$$

where the asymptotic parameters are $(h_1, h_{11}, h_2, f_1, f_2)$. Note that all logarithms disappear when h_{11} is set to zero.

The mass, angular momentum and charge may be extracted from the asymptotic behavior of the soliton. In terms of the five parameters given above, we find

$$
M = \frac{1}{4}(h_1 - 3f_1 + 2f_2) - \frac{1}{4}h_{11}\left(5 + \frac{13}{6}h_1 - 3f_1 - h_{11}\left(\frac{73}{12} + h_1 - 2h_{11}\right) - \frac{1}{3}h_{11}\log x\right),
$$
$$
gJ = \frac{1}{4}(-f_1 + f_2) - \frac{1}{8}h_{11}(-3f_1 + (2 - h_{11})(2 + h_1 - 2h_{11})),
$$
$$
Q_1 = \frac{1}{4}(h_1 - f_1) - \frac{1}{4}h_{11}\left(1 + \frac{1}{2}h_1 - \frac{1}{2}h_{11}\right). \tag{6.12}
$$

Note the $\log x$ term in the expression for the mass, which arises because the mass is obtained from the asymptotic form of the metric in the limit $x \to \infty$. Clearly this indicates that the mass is divergent, except in the case $h_{11} = 0$. This divergence also shows up in the modified BPS expression

$$
M = 2gJ + Q_1 + \frac{1}{12}h_{11}\left(h_1 - \frac{5}{4}h_{11} + h_{11}\log x\right). \tag{6.13}
$$

While at first sight this divergence may appear surprising, there is in fact a natural explanation for where it arises. At infinity, constant time slices of the five-dimensional metric take the form

$$
ds^2 = H^{1/3}\left(\frac{dx^2}{4xh} + \frac{x}{4}\left(\sigma_1^2 + \sigma_2^2 + \left(h - \frac{4w_3^2}{xH}\right)\sigma_3^2\right)\right)
$$
$$
\sim \frac{dx^2}{4g^2x^2} + \frac{x}{4}\left(\sigma_1^2 + \sigma_2^2 + \left(1 + \frac{1}{2}h_{11}\right)\sigma_3^2\right). \tag{6.14}
$$

As a result, h_{11} parameterizes the distortion of the S^3 at infinity. The reason for the divergent mass is simply that the space is no longer asymptotically AdS_5 whenever $h_{11} \neq 0$. This squashing of the S^3 is generated by a constant magnetic field at infinity

$$
F^1 \sim -\frac{h_{11}}{2g}\sigma_1 \wedge \sigma_2. \tag{6.15}
$$

Turning now to the origin, demanding regularity of the soliton at $x = 0$ yields a two-parameter family of solutions specified by (h_0, ζ_0):

$$
H = h_0 + \frac{g^2 x}{2}(1 - h_0^2 - 6\zeta_0(h_0 + 3\zeta_0)) + \cdots,
$$
$$
\zeta = \zeta_0 x - \frac{g^2 x^2}{2}\zeta_0(h_1 + 3\zeta_0) + \cdots. \tag{6.16}
$$

Note that the previous relation $\cosh\varphi_1 = (xH_1)'$ is recovered in the limit $\zeta_0 = 0$. The matching of this expansion at $x = 0$ to the asymptotic one (6.12) appears nontrivial but can nevertheless be approached numerically. We find that h_{11} vanishes only when $\zeta_0 = 0$. Moreover, the angular momentum J in (6.12) also vanishes only when $\zeta_0 = 0$.

Turning off ζ_0 yields the regular one-charge bubble of [10], with function H_1 given by (6.4) (and with $c = 1$). In general, h_0 is related to the R-charge, while ζ_0 is related to the rotation. In this one-charge case, non-zero rotation generates a magnetic field at infinity, resulting in a squashing of S^3 and hence a divergent mass expression. We have also examined the three-equal charge soliton, where we found similar behavior, except that the mass expression remains finite, even with the magnetic field and squashing (parameterized by the analog of h_{11}) present.

The second type of solitonic solutions can be referred to as $\mathbb{R}^2$ solitons, for which $0 < x_0 \leq x < \infty$. In this case, the geometry at $x = x_0$ is a timelike bundle over $\mathbb{R}^2 \times S^2$. As in the first type of soliton, we can perform a Taylor expansion around $x = x_0$, for which H_I and φ_I are constants at the zeroth order, and $h(x)$, $w_3(x)$ and $U_3^I(x)$ vanish linearly when x approaches x_0. We used numerical methods to demonstrate that, for appropriately chosen parameters, there are solutions for which x runs smoothly from x_0 to ∞, with no CTC's.

7. Bubble generalizations of Klemm-Sabra solutions

Since we have focused on an $\mathcal{N} = 2$ truncation of the full $\mathcal{N} = 8$ theory, some care must be taken when counting the total number of preserved supersymmetries. From an $\mathcal{N} = 8$ perspective, the general BPS bound has the form

$$M \geq \pm gJ_1 \pm gJ_2 \pm Q_1 \pm Q_2 \pm Q_3, \tag{7.1}$$

where an even number of minus signs are to be taken. For two generic angular momenta and three generic charges, saturation of this bound holds for only a single choice of signs. Thus, generic three-charge solutions with two independent rotations preserve two real supersymmetries out of 32 (i.e., they are 1/16 BPS in $\mathcal{N} = 8$).

The $\mathcal{N} = 2$ truncation that we have taken in section 2.2, with $\mathcal{N} = 2$ graviphoton given by (2.25), yields a BPS bound with correlated signs for the R-charges:

$$M \geq \pm gJ_1 \pm gJ_2 \pm (Q_1 + Q_2 + Q_3). \tag{7.2}$$

(Again, we take an even number of minus signs.) Generic rotating black holes then preserve 1/4 of the $\mathcal{N} = 2$ supersymmetries, or two real supersymmetries out of 8, in agreement with the $\mathcal{N} = 8$ analysis.

By focusing on a cohomogeneity one base with bi-axial symmetry, we have essentially set the two angular momenta J_1 and J_2 equal to each other ($J_1 = J_2 = J$). In this case, the reduced BPS condition becomes

$$M \geq \begin{cases} 2gJ + Q_1 + Q_2 + Q_3 \\ -2gJ + Q_1 + Q_2 + Q_3 \\ - Q_1 - Q_2 - Q_3 \\ - Q_1 - Q_2 - Q_3. \end{cases} \tag{7.3}$$

The solutions that we have examined above saturate the first line of the BPS bound, as can be seen from (5.12) for the family of solutions without hypermatter scalars. These solutions generically preserve 1/4 of the $\mathcal{N} = 2$ supersymmetries, as was explicitly demonstrated by constructing the projection (3.27) out of the hyperino variations.

It should be noted that saturation of the BPS conditions (7.3) may also be achieved by taking

$$M = -(Q_1 + Q_2 + Q_3).\qquad(7.4)$$

This gives rise to a second independent class of solutions preserving 1/2 of the $\mathcal{N} = 2$ supersymmetries. In fact, this family of solutions was originally constructed by Klemm and Sabra in [33, 34] using a variety of methods including formal analytic continuation. Furthermore, we find that this can be generalized by turning on the hypermatter scalars φ_I. The result is given by

$$ds^2 = -(H_1 H_2 H_3)^{-2/3}(dt + w_3\sigma_3)^2 + (H_1 H_2 H_3)^{1/3}\left(\frac{dx^2}{4xh} + \frac{x}{4}(\sigma_1^2 + \sigma_2^2 + h\sigma_3^2)\right),$$

$$A^I = \frac{1}{H_I}(dt + w_3\sigma_3) - U_3^I\sigma_3, \qquad X^I = (H_1 H_2 H_3)^{1/3}/H_I, \qquad \cosh\varphi_I = (xH_I)',\text{(7.5)}$$

where

$$w_3 = \frac{1}{2}\left(gx\,H_1 H_2 H_3 - \frac{\alpha}{x}\right),$$

$$h = 1 - \frac{2g\,\alpha}{x} + g^2 x\,H_1 H_2 H_3,$$

$$U_3^I = \frac{g}{2}x\,H_J H_K,\qquad(7.6)$$

and where the functions H_I obey the equation (6.2). As before, the single-charge case has an explicit solution given by (6.4).

While this Klemm-Sabra generalization is written in a similar form to that implied by the supersymmetry analysis of section 3, it has an important difference in that the sign of the gauge potential in (7.5) is *opposite* to that of (3.35). This suggests that the Klemm-Sabra solution does not fall into the same class as those satisfying the supersymmetry construction of section 4, a situation which was already hinted at in [15]. In fact, it is easy to verify that the Klemm-Sabra functions (7.6) do not satisfy the relevant set of first-order equations (4.7) found above, thus explicitly demonstrating the incompatibility of the Klemm-Sabra solution with the construction of section 4.

Although this incompatibility might appear to demonstrate a flaw in the supersymmetry analysis of section 3 (which purports to capture *all* supersymmetric solutions), this is actually not the case. The reason for this is that the Klemm-Sabra family of 1/2 BPS solutions saturates the last two lines of the BPS bound in (7.3), in contrast to the 1/4 BPS solutions which instead saturate the first. As a result, the Klemm-Sabra Killing spinors have a different nature from the ones constructed above in section 3.4. With a different Killing spinor, the invariant tensors (3.1) are modified, and in particular the preferred Killing vector $K^\mu = \frac{1}{2}\bar{\epsilon}^i\gamma^\mu\epsilon_i$ is no longer of the form $\partial/\partial t$ for the Klemm-Sabra solution given here. This indicates that, while the spatial slices of the metric (7.5) have the same

cohomogeneity-one form as (4.1), this metric is not the preferred Kähler metric h_{mn} of the base given in (3.6). Essentially, the Klemm-Sabra solution as written here has not been put into the preferred coordinate system implied by (3.6), despite the superficial similarities.

This difference in Killing spinors can be demonstrated more explicitly by first considering the maximally symmetric AdS$_5$ vacuum written as

$$ds^2 = -\left(dt + \frac{1}{2}gx\sigma_3\right)^2 + \frac{dx^2}{4xh} + \frac{x}{4}(\sigma_1^2 + \sigma_2^2 + h\sigma_3^2), \tag{7.7}$$

where $h = 1 + g^2x$. Noting that spinors on AdS$_5$ transform as $(\mathbf{2},\mathbf{1}) + (\mathbf{1},\mathbf{2})$ under $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$, we find that the Killing spinors corresponding to the gravitino variation of (2.24) decompose as $\mathbf{1} + \mathbf{1} + \mathbf{2}$ under $\mathrm{SU}(2)_R$. To see this explicitly, it is helpful to adopt a complex spinor notation, in which any symplectic Majorana spinor pair, say ψ_i, is regrouped as a complex spinor $\psi \equiv \psi_1 + i\psi_2$. In this case, using the vierbein basis of (4.2) along with $e^0 = dt + \frac{1}{2}gx\sigma_3$, we introduce constant complex spinors $\chi_0^{\pm\pm}$ satisfying the mutually commuting projections

$$\gamma^{23}\chi_0^{\pm\alpha} = \mp i\chi_0^{\pm\alpha}, \qquad \gamma^{14}\chi_0^{\alpha\pm} = \pm i\chi_0^{\alpha\pm}, \tag{7.8}$$

where $\alpha = \pm$. Using the convention that $\gamma^{01234} = i$, the above projections are compatible with

$$i\gamma^0\chi_0^{++} = -\chi_0^{++}, \qquad i\gamma^0\chi_0^{--} = -\chi_0^{--}, \tag{7.9}$$

as well as

$$i\gamma^0\chi_0^{+-} = \chi_0^{+-}, \qquad i\gamma^0\chi_0^{-+} = \chi_0^{-+}. \tag{7.10}$$

Note that (7.9) is compatible with the projection found above in (3.25). The Killing spinors of AdS$_5$ are then comprised of the two singlets

$$\epsilon^{(\mathbf{1})} = e^{\frac{3}{2}igt}\chi_0^{++},$$
$$\epsilon^{(\mathbf{1}')} = e^{-\frac{1}{2}igt}\left[\sqrt{g^2x} - \gamma^1\sqrt{h}\right]\chi_0^{-+}, \tag{7.11}$$

as well as the doublet

$$\epsilon^{(\mathbf{2})} = e^{-\frac{1}{2}igt}\left(\gamma^3 u + [\sqrt{g^2x} - \gamma^1\sqrt{h}]v\right)\chi_0^{+-}. \tag{7.12}$$

The functions u and v are given on the SU(2) orbits by

$$\begin{pmatrix} u \\ v \end{pmatrix} = U^{-1}\begin{pmatrix} u_0 \\ v_0 \end{pmatrix}, \tag{7.13}$$

where u_0 and v_0 are arbitrary constants and the SU(2) matrix U parameterizes the orbits. In terms of Euler angles (θ, ϕ, ψ), U may be written as

$$U = e^{J_3\phi}e^{J_2\theta}e^{J_3(\psi+\frac{1}{2}\pi)}, \tag{7.14}$$

where $J_i = -\frac{i}{2}\tau_i$ with τ_i being the standard Pauli matrices. The left-invariant one-forms σ_i which show up in (7.7) are given by $U^{-1}dU = J_i\sigma_i$.

Before turning on rotation, we consider the stationary BPS superstar configuration of [2, 3]. The canonical form of this solution may be obtained by setting $\gamma_I = 0$ in (5.7). In this case, the superstar field configuration takes the form

$$ds^2 = -\mathcal{H}^{-2/3}(dt + w_3\sigma_3)^2 + \mathcal{H}^{1/3}\left(\frac{dx^2}{4xh} + \frac{x}{4}(\sigma_1^2 + \sigma_2^2 + h\sigma_3^2)\right),$$

$$A^I = -\frac{1}{H_I}(dt + w_3\sigma_3) + U_3^I\sigma_3, \qquad X^I = \mathcal{H}^{1/3}/H_I, \tag{7.15}$$

where

$$w_3 = \frac{1}{2}gx\mathcal{H}, \qquad h = 1 + g^2x\mathcal{H}, \qquad U_3^I = \frac{1}{2}gx\mathcal{H}/H_I, \tag{7.16}$$

and

$$H_I = 1 + \frac{q_I}{x}, \qquad \mathcal{H} = H_1H_2H_3. \tag{7.17}$$

This configuration breaks half of the supersymmetries, and the surviving Killing spinors have the form

$$\epsilon^{(\mathbf{1})} = \mathcal{H}^{-1/6}\chi_0^{++},$$

$$\epsilon^{(\mathbf{1'})} = \mathcal{H}^{-1/6}e^{-2igt}\left[\sqrt{g^2x\mathcal{H}} - \gamma^1\sqrt{h}\right]\chi_0^{-+}. \tag{7.18}$$

These Killing spinors clearly generalize the singlet AdS_5 Killing spinors (7.11). (Note that the modified time dependence is related to the turning on of a constant gauge potential at infinity, $A^I(\infty) = -dt$.) After rotation is included to obtain Gutowksi-Reall black holes [5, 6], then only $\epsilon^{(\mathbf{1})}$ survives as a Killing spinor, in agreement with the projections (3.25) and (3.27) found above.

Taking $\epsilon^{(\mathbf{1})}$ as the preferred Killing spinor, we may verify that the $Sp(2)$ singlet bilinears take on the form

$$f = \frac{i}{2}\bar{\epsilon}^{(\mathbf{1})}\epsilon^{(\mathbf{1})} = \mathcal{H}^{-1/3},$$

$$K^\alpha = \frac{1}{2}\bar{\epsilon}^{(\mathbf{1})}\gamma^\alpha\epsilon^{(\mathbf{1})} = \mathcal{H}^{-1/3}[1, 0, 0, 0, 0], \tag{7.19}$$

where we have normalized the constant spinor χ_0^{++} according to $\frac{i}{2}\overline{\chi}_0^{++}\chi_0^{++} = 1$. This indicates that the preferred Killing vector $K^\mu\partial_\mu = \partial/\partial t$ has indeed been chosen properly to agree with the metric decomposition chosen in (3.6).

In contrast to the Gutowksi-Reall black holes, the generalized Klemm-Sabra solution (7.5) with (7.6) preserve the opposite set of Killing spinors, given by the doublet

$$\epsilon^{(\mathbf{2})} = \mathcal{H}^{-1/6}e^{igt}\left(\gamma^3 u + [\sqrt{g^2x\mathcal{H}} - \gamma^1\sqrt{h}]v\right)\chi_0^{+-}, \tag{7.20}$$

where $h = 1 + g^2x\mathcal{H} - 2g\alpha/x$, and u and v are again given by (7.13). In this case, the relevant spinor bilinears take on a more complex form

$$f = \frac{i}{2}\bar{\epsilon}^{(\mathbf{2})}\epsilon^{(\mathbf{2})} = \mathcal{H}^{-1/3}\left(1 - \frac{2g\alpha}{x}|v|^2\right),$$

$$K^\alpha = \frac{1}{2}\bar{\epsilon}^{(\mathbf{2})}\gamma^\alpha\epsilon^{(\mathbf{2})} = \mathcal{H}^{-1/3}\left[1 + \left(2g^2x\mathcal{H} - \frac{2g\alpha}{x}\right)|v|^2, 0,\right.$$

$$\left. 2\sqrt{g^2x\mathcal{H}}\,\mathfrak{Re}(u^*v), 2\sqrt{g^2x\mathcal{H}}\,\mathfrak{Im}(u^*v), 2\sqrt{h}\sqrt{g^2x\mathcal{H}}|v|^2\right], \tag{7.21}$$

where now our normalization is given by $\frac{i}{2}\overline{\chi}_0^{+-}\chi_0^{+-} = -1$ and $|u|^2 + |v|^2 = 1$. Although the normalization relation $K^2 = -f^2$ continues to hold (as it must by construction), this time the preferred timelike Killing vector K^μ no longer points simply along $\partial/\partial t$. In fact, even in the absence of rotation ($\alpha = 0$), the Killing vector still has components along the four-dimensional base. As a result, we see that the Klemm-Sabra solution, as given by (7.5), is not in the canonical form (3.6) for the supersymmetry analysis, despite superficial appearances.

Of course, it is possible to perform an appropriate coordinate transformation to put the Klemm-Sabra solution into canonical form. Doing so, however, will break manifest $\mathrm{SU}(2)_L \times \mathrm{U}(1)$ invariance to $\mathrm{U}(1)^2$, corresponding to introducing a cohomogeneity-two base appropriate to the turning on of two independent rotations. The Klemm-Sabra solution is then recovered in the limit when $J_1 = J_2$ (in which case the full $\mathrm{SU}(2)_L \times \mathrm{U}(1)$ isometry as well as cohomogeneity-one gets restored as a hidden symmetry).

Finally, we wish to make the observation that the Killing spinors $\epsilon^{(1)}$, $\epsilon^{(1')}$ and $\epsilon^{(2)}$ (if they exist) may be put in one-to-one correspondence with the first, second and last two lines of the BPS inequalities given in (7.3).

8. Conclusions

We have used the G-structure approach to construct supersymmetric solutions of five-dimensional $\mathcal{N} = 2$ gauged supergravity coupled to two vector multiplets and three incomplete hypermultiplets, which arises from a truncation of five-dimensional $\mathcal{N} = 8$ gauged supergravity. Different types of previously-known supersymmetric solutions arise within this unified framework, including rotating black holes, AdS bubbles, solitons and time machines. New families of rotating AdS bubbles and solitonic solutions are presented.

In addition, there are some rather exotic solitons without hyperscalars. These include 'texture'-like zero mass solitons and 'phantom'-like negative-mass solitons, where the mass is defined relative to the AdS vacuum. These constitute explicit examples of negative mass supergravity solutions which are completely regular and free of closed timelike curves. In addition, being supersymmetric guarantees that they are perturbatively free of local instabilities. These solutions evade positive mass theorems for asymptotically AdS solutions by being asymptotically $\mathrm{AdS}_5/\mathbb{Z}_k$ with $k \geq 3$. In particular, the S^3 within AdS_5 has been replaced by the lens space $S^3/\mathbb{Z}_k$.

It would be interesting to investigate how these exotic solitons can be interpreted in terms of the AdS/CFT correspondence. In particular, what is the physical quantity in the dual field theory that corresponds to the negative mass? Determining whether the field theory undergoes runaway behavior could offer some insight into the physical nature of these solitons.

One could also see if asymptotically locally flat or de Sitter solitons with negative mass could be constructed in four-dimensional theories with zero or positive cosmological constant, and whether they share some of the properties of the solitons discussed in this paper. In particular, the negative mass solitons with three equal charges do not have any scalar fields turned on. Thus, analogous solutions might exist in four-dimensional Einstein-

Maxwell de Sitter gravity. Due to their repulsive behavior, such solitons might be used to model the observed acceleration of the scale factor $a(t)$ of the universe [31, 32].

The construction of supersymmetric solutions relies upon a choice of a four-dimensional Kähler base. We have limited ourselves to a cohomogeneity-one base with bi-axial symmetry, which preserves $SU(2)_L \times U(1) \subset SU(2)_L \times SU(2)_R \simeq SO(4)$ isometry. This case encompasses all known black holes and AdS bubbles with two equal rotations turned on. The more general case of two unequal rotations would require a cohomogeneity-two base. On a similar note, one could also consider a tri-axial four-dimensional Kähler base space. The corresponding system of equations is presented in appendix B, though no solutions are known except for a couple of special cases.

We have only considered the first-order equations for supersymmetric backgrounds that preserve a time-like Killing vector. One could also consider supersymmetric systems with a null Killing vector. An example of such a solution in which the hyperscalars have not been turned on is the magnetic string of [35], which was shown in [15] to preserve a null Killing vector.

Lastly, there are a number of other possible generalizations of the solutions discussed in this paper, such as analogous constructions in different dimensions as well as non-supersymmetric generalizations, e.g., non-extremal rotating black holes in gauged supergravities [29, 30, 36, 7, 37]. Non-extremal static AdS bubbles were explored in [38]. It would be interesting to investigate whether there is a non-extremal generalization that includes both the rotating black hole as well the rotating AdS bubble.

Acknowledgments

A. Differential identities for the spinor bilinears

In this appendix, we present the differential identities arising from the gravitino, hyperino and gaugino transformations. We first present the raw identities, and then rewrite them in a more suggestive form notation.

The differential identities arising from the gravitino variation are

$$
\begin{aligned}
\partial_\mu f &= \frac{1}{3} i_K \mathcal{F}, \\
\nabla_\mu K_\nu &= \frac{1}{3} f \mathcal{F}_{\mu\nu} + \frac{1}{12} \epsilon_{\mu\nu}{}^{\rho\lambda\sigma} \mathcal{F}_{\rho\lambda} K_\sigma - \frac{1}{3} W \Phi^2_{\mu\nu}, \\
\nabla_\mu \Phi^a_{\nu\lambda} &= \frac{1}{6} \left(-g_{\mu[\nu} \mathcal{F}^{\alpha\beta} * \Phi^a_{\lambda]\alpha\beta} + 2\mathcal{F}_{[\nu}{}^\alpha * \Phi^a_{\lambda]\mu\alpha} - 2\mathcal{F}_\mu{}^\alpha * \Phi^a_{\nu\lambda\alpha} \right) \\
&\quad - \frac{2}{3} W \delta^{a2} g_{\mu[\nu} K_{\lambda]} - \epsilon^{2ab} \left(g\mathcal{A}_\mu \Phi^b_{\nu\lambda} - \frac{1}{3} W * \Phi^b_{\mu\nu\lambda} \right).
\end{aligned}
\tag{A.1}
$$

Note that the gauging explicitly breaks the $Sp(2)$ symmetry.

The hyperino transformation $\delta\lambda_{I\,i}$ from (2.24) gives rise to

$$i_K d\varphi_I = 0,$$
$$f\partial_\mu\varphi_I = -2g\sinh\varphi_I \Phi^2_{\mu\nu}A^I_\nu,$$
$$2K_{[\nu}\partial_{\mu]}\varphi_I = 2g\sinh\varphi_I(A^I_\lambda * \Phi^2_{\mu\nu\lambda} + X^I\Phi^2_{\mu\nu}),$$
$$*\Phi^a_{\mu\nu\lambda}\partial_\lambda\varphi_I = 2g\sinh\varphi_I[\epsilon^{2ab}(*\Phi^b_{\mu\nu\lambda}A^I_\lambda + X^I\Phi^b_{\mu\nu}) - 2\delta^{a2}A^I_{[\mu}K_{\nu]}],$$
$$\Phi^a_{\mu\nu}\partial_\nu\varphi_I = 2g\sinh\varphi_I[\epsilon^{2ab}\Phi^b_{\mu\nu}A^I_\nu + \delta^{a2}(fA^I_\mu - K_\mu X^I)],$$
$$g\sinh\varphi_I(i_K A^I + fX^I) = 0. \tag{A.2}$$

Note that, provided $g\sinh\varphi_I \neq 0$, we have the condition

$$i_K A^I = -fX^I, \tag{A.3}$$

relating the electric potential to the scalars. (Recall that gauge invariance is lost when $g\varphi_I$ is turned on.) Even for $g\sinh\varphi_I = 0$, we may take this as a gauge condition.

From the gaugino transformations $\delta\chi^{(\alpha)}_i$, we have

$$i_K d\phi^{(\alpha)} = 0,$$
$$f d\phi^{(\alpha)} = i_K F^{(\alpha)},$$
$$2K_{[\nu}\partial_{\mu]}\phi^{(\alpha)} = [fF^{(\alpha)}_{\mu\nu} - (i_K * F^{(\alpha)})_{\mu\nu}] + 2g\partial_\alpha\hat{W}\Phi^2_{\mu\nu},$$
$$*\Phi^a_{\mu\nu\lambda}\partial^\lambda\phi^{(\alpha)} = -2F^{(\alpha)\lambda}_{[\mu}\Phi^a_{\nu]\lambda} + g\epsilon^{2ab}\partial_\alpha\hat{W}\Phi^b_{\mu\nu},$$
$$\Phi^a_{\mu\nu}\partial^\nu\phi^{(\alpha)} = -\frac{1}{4}\epsilon_{\mu\nu\lambda\rho\sigma}F^{(\alpha)}_{\nu\lambda}\Phi^a_{\rho\sigma} - 2g\delta^{a2}\partial_\alpha\hat{W}K_\mu,$$
$$\Phi^a_{\mu\nu}F^{(\alpha)\,\mu\nu} = -4g\delta^{a2}f\partial_\alpha\hat{W}, \tag{A.4}$$

where

$$\phi^{(1)} = 3\log(X^1),$$
$$F^{(1)}_{\mu\nu} = \frac{2}{X^1}F^1_{\mu\nu} - \frac{1}{X^2}F^2_{\mu\nu} - \frac{1}{X^3}F^3_{\mu\nu},$$
$$\partial_1\hat{W} = 2X^1\cosh\varphi_1 - X^2\cosh\varphi_2 - X^3\cosh\varphi_3, \tag{A.5}$$

and similarly for $\alpha = 2$.

The above identities can be put into form notation. The gaugino and gravitino differential identities combine nicely to yield the 0-form identities

$$i_K dX^I = 0,$$
$$\Phi^{a\,\mu\nu}\left(\frac{2}{X^1}F^1_{\mu\nu} - \frac{1}{X^2}F^2_{\mu\nu} - \frac{1}{X^3}F^3_{\mu\nu}\right) = -4g\delta^{a2}f(2X^1\cosh\varphi_1 - X^2\cosh\varphi_2 - X^3\cosh\varphi_3),$$
$$\Phi^{a\,\mu\nu}\left(-\frac{1}{X^1}F^1_{\mu\nu} + \frac{2}{X^2}F^2_{\mu\nu} - \frac{1}{X^3}F^3_{\mu\nu}\right) = -4g\delta^{a2}f(-X^1\cosh\varphi_1 + 2X^2\cosh\varphi_2 - X^3\cosh\varphi_3),$$
$$\tag{A.6}$$

the 1-form identities

$$d(fX^I) = i_K F^I, \tag{A.7}$$

the 2-form identities

$$d\left(\frac{1}{X^I}K\right) = i_K * \left(\frac{1}{(X^I)^2}F^I\right) + f(X^J F^K + X^K F^J) - 2g\Phi^2\cosh\varphi_I, \tag{A.8}$$

(where $I \neq J \neq K$) the 3-form identities

$$(d\delta^{ab} + g\epsilon^{2ab}\mathcal{A}\wedge)\Phi^b = \epsilon^{2ab}W * \Phi^b,$$
$$\Phi^a \wedge d\log X^1 = -\frac{1}{3}F_\mu^{(1)\,\sigma} * \Phi^a_{\nu\lambda\sigma}\frac{1}{2}dx^\mu \wedge dx^\nu \wedge dx^\lambda - \frac{1}{3}g\epsilon^{2ab}\partial_1\hat{W} * \Phi^b,$$
$$\Phi^a \wedge d\log X^2 = -\frac{1}{3}F_\mu^{(2)\,\sigma} * \Phi^a_{\nu\lambda\sigma}\frac{1}{2}dx^\mu \wedge dx^\nu \wedge dx^\lambda - \frac{1}{3}g\epsilon^{2ab}\partial_2\hat{W} * \Phi^b, \tag{A.9}$$

and the 4-form identities

$$(d\delta^{ab} + g\epsilon^{2ab}\mathcal{A}\wedge) * (X^I\Phi^b) = F^I \wedge \Phi^a + 2g\delta^{a2}\left(\frac{1}{X^J}\cosh\varphi_K + \frac{1}{X^K}\cosh\varphi_J\right) * K. \tag{A.10}$$

The hyperinos add the following:

$$i_K d\varphi_I = 0,$$
$$i_K A^I = -fX^I,$$
$$f d\varphi_I = -2g\sinh\varphi_I \Phi^2_{\mu\nu}A^{I\,\nu}dx^\mu,$$
$$d\varphi_I \wedge K = 2g\sinh\varphi_I\left[X^I\Phi^2 + *\Phi^2_{\mu\nu\lambda}A^{I\,\lambda}\frac{1}{2}dx^\mu \wedge dx^\nu\right],$$
$$\Phi^a \wedge d\varphi_I = 2g\sinh\varphi_I[\epsilon^{2ab}(\Phi^b \wedge A^I - X^I * \Phi^b) + \delta^{a2} * (A^I \wedge K)],$$
$$*\Phi^a \wedge d\varphi_I = 2g\sinh\varphi_I[\epsilon^{2ab} * \Phi^b \wedge A^I - \delta^{a2} * (fA^I - KX^I)]. \tag{A.11}$$

B. The tri-axial case

A more general tri-axial class of cohomogeneity-one solutions with S^3 orbits has the following ansatz for the four-dimensional Kähler base:

$$ds_4^2 = h^2\,dx^2 + \sum_{i=1}^3 a_i^2\sigma_i^2\,, \tag{B.1}$$

where the functions h and a_i depend on x only, and the σ_i are left-invariant one-forms satisfying $d\sigma_i = -\frac{1}{2}\epsilon^{ijk}\sigma_j \wedge \sigma_k$. We also introduce a natural vielbein basis

$$e^0 = h\,dx\,, \qquad e^i = a_i\,\sigma_i\,. \tag{B.2}$$

An ansatz for the SU(2) invariant anti-self-dual Kähler form is

$$J = \sum_{i=1}^3 \alpha_i(x)\left(e^0 \wedge e^i - \frac{1}{2}\epsilon_{ijk}e^j \wedge e^k\right), \tag{B.3}$$

where $\sum_i \alpha_i^2 = 1$. The base metric is Kähler if J is covariantly constant, which implies that

$$\alpha_1' = \left(\frac{h(a_3^2 - a_1^2 - a_2^2)}{2a_1a_2a_3} - \frac{a_3'}{a_3}\right)\alpha_2 + \left(\frac{a_2'}{a_2} - \frac{h(a_2^2 - a_1^2 - a_3^2)}{2a_1a_2a_3}\right)\alpha_3, \qquad \text{and cyclic}, \quad \text{(B.4)}$$

where a prime denotes a derivative with respect to x. For simplicity, we will take $\alpha_1 = \alpha_2 = 0$ and $\alpha_3 = 1$, so that

$$J = e^0 \wedge e^3 - e^1 \wedge e^2, \tag{B.5}$$

and

$$\begin{aligned} 2a_1'a_2a_3 &= h(a_2^2 + a_3^2 - a_1^2), \\ 2a_1a_2'a_3 &= h(a_1^2 + a_3^2 - a_2^2). \end{aligned} \tag{B.6}$$

Note that these two above conditions imply that

$$h\,a_3 = (a_1\,a_2)'. \tag{B.7}$$

The base also has the Ricci form

$$\begin{aligned} \mathcal{R} &= \frac{1}{2a_1a_2a_3}\left[\frac{a_3^4 - (a_1^2 - a_2^2)^2}{a_1a_2a_3} - \frac{2a_1a_2a_3''}{h} + \frac{2a_1a_2a_3'h'}{h^3} - \frac{2a_3a_3'}{h}\right]e^0 \wedge e^3 \\ &\quad + \frac{2a_1a_2a_3' + (a_3^2 - a_1^2 - a_2^2)h}{2a_1^2a_2^2h}\,e^1 \wedge e^2, \end{aligned} \tag{B.8}$$

which can be expressed as

$$\mathcal{R} = d\left[\left(\frac{a_1^2 + a_2^2 - a_3^2}{2a_1a_2} - \frac{a_3'}{h}\right)\sigma_3\right]. \tag{B.9}$$

We make the following ansatz for the one-forms ω and β^I:

$$\omega = \sum_{i=1}^{3} w_i\sigma_i, \qquad \beta^I = \sum_{i=1}^{3} U_i^I\sigma_i. \tag{B.10}$$

$d\omega$ decomposes into self-dual and anti-self dual components according to

$$(d\omega)^{\pm} = \frac{1}{2}\sum_{i \neq j \neq k}\left(\frac{w_i'}{ha_i} \mp \frac{w_i}{a_ja_k}\right)\left(e^0 \wedge e^i \pm \frac{1}{2}\epsilon^{ijk}e^j \wedge e^k\right). \tag{B.11}$$

$(d\beta^I)^{\pm}$ have the same form as $(d\omega)^{\pm}$, except with $w_i \to U_i^I$.

Inserting these expressions into the supersymmetry conditions (3.37) gives rise to the first-order equations

$$\varphi_I' = -2g\frac{hU_3^I}{a_3}\sinh\varphi_I,$$

$$\frac{U_3^{I'}}{ha_3} + \frac{U_3^I}{a_1a_2} = 2g(H_J\cosh\varphi_K + H_K\cosh\varphi_J), \qquad I \neq J \neq K$$

$$U_1^{I'} = -\frac{ha_1}{a_2a_3}U_1^I, \qquad U_2^{I'} = -\frac{ha_2}{a_1a_3}U_2^I,$$

$$w_i' - \frac{ha_i}{a_ja_k}w_i = \frac{1}{2}\sum_I H_I\left(U_i^{I'} - \frac{ha_i}{a_ja_k}U_i^I\right), \qquad i \neq j \neq k$$

$$\frac{a_3^2 - a_1^2 - a_2^2}{2a_1a_2} + \frac{a_3'}{h} = g\sum_I U_3^I\cosh\varphi_I, \tag{B.12}$$

as well as the algebraic conditions

$$\sum_I U_\ell^I\cosh\varphi_I = 0, \qquad gU_\ell^I\sinh\varphi_I = 0, \tag{B.13}$$

where $\ell = 1, 2$.

In addition, the second-order equation of motion (3.38) reduces to

$$0 = \left(\frac{a_1a_2a_3}{h}H_I' + \sum_i U_i^J U_i^K\right)' - 2g\cosh\varphi_I(a_1a_2w_3' + ha_3w_3) + 4g^2ha_1a_2a_3\sinh^2\varphi_I x H_J H_K. \tag{B.14}$$

Consider the purely gravitational system with $\varphi_I = U_1^I = U_2^I = 0$, $H_I = 1$, for which

$$U_3^I = w_3 = 2ga_1a_2,$$
$$w_\ell = c_\ell\exp\left[\int^x dx\,\frac{ha_3}{a_1a_2}\right]. \tag{B.15}$$

Then the base space is Einstein-Kähler, and is described by the functions a_i which obey the equations (B.6) along with

$$2a_1a_2a_3' = (a_1^2 + a_2^2 - a_3^2 + 12g^2a_1^2a_2^2), \tag{B.16}$$

where we have chosen a gauge such that $h = 1$. This system of equations has been considered by Dancer and Strachan [39]. Explicit solutions are only known for comparatively simple examples such as the tri-axial forms of the Fubini-Study metric on $\mathbb{CP}^2$ and the product metric on $\mathbb{CP}^1 \times \mathbb{CP}^1$. Since this does not bode well for finding explicit solutions with additional fields, in this paper we have focused on the bi-axial case $a_1 = a_2$.

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